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Positivity of Bondi Mass in Odd Dimensions

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Abstract

We proof that the Bondi mass of an asymptotically flat, vacuum spacetime is non-negative in all odd dimensions $d \geq 5$ assuming that a suitable spinor fulfilling the Witten equation exists. This extends classical results by Witten and others on the positivity in four dimensions and recent results by Hollands and Thorne in even higher dimensions. Our proof holds for manifolds which admit Killing spinors near infinity, in particular, if infinity is the standard sphere. To enable our proof we investigate the asymptotic expansion of Bondi coordinates and how imposing Einstein equations restricts the allowed asymptotic decay of metric coefficients and spinor fields. We also derive a coordinate expression for the Bondi mass in odd dimensions.

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Contents

CHAPTER 1

Introduction

It would probably be surprising to see that a theory of fundamental importance to the understanding of nature and the foundations of physics is largely ignored for half a century. However, this was the case for general relativity which, after its conception around 1915, did not receive much attention. It was only around the year of Einstein's death in 1955 that general relativity attracted more attention and entered the mainstream of theoretical physics. This shift in attitude was due to several breakthroughs, and the period from the 1960s to the mid 1970s has been coined the "golden age of general relativity" [1]. Several of the concepts, which are now paradigmatic for general relativity, were found during this time. R. Kerr found the Kerr metric (uncharged, rotating black hole) in 1963 [2], the no-hair conjecture was stated and some simple cases proven, e.g., Israel showed the uniqueness of the Schwarzschild metric in 1967 [3]. The first few of the famous singularity theorems were proven by R. Penrose and S. Hawking between 1965 and 1970 [4–6]. The foundation of black hole thermodynamics was laid in 1973 by J. Bekenstein [7] and J. Bardeen, B. Carter and S. Hawking [8] and in 1975 Hawking showed the existence of Hawking radiation [9] just one year after the first indirect evidence of gravitational waves was discovered by R. Hulse and J. Taylor in 1974. In the same period several other authors worked on a problem of tantamount importance, but which has received much less publicity due to its technical nature. It was only starting in 1960 that a rigorous definition of mass was available [10, 11]. At the first glance, this problem seems to be either irrelevant or trivial. After all, not much thought is spend on this in other theories and it is not obvious that one should do anything differently in general relativity. So why was this considered a relevant/non-trivial problem? The reason lies at the very foundation of general relativity and is, in fact, the argument that lead Einstein 1907 to his theory of gravitation, namely, the equivalence principle. It states that the trajectory of a point mass in a gravitational field is independent of its structure (in vacuum, a rock and a feather fall in the same way) and that the outcome of an experiment is independent of the velocity and position of the laboratory. Another way of saying this is that in a closed laboratory (no interactions with the environment) it is not possible to distinguish between a system which accelerates far away from any masses and a system which is in free fall close to a mass. In Einsteins words:

> "Wir betrachten zwei Bezugssysteme Σ_1 und Σ_2. Σ_1 sei in Richtung der X-Achse beschleunigt, und es sei γ die (zeitlich konstante) Größe dieser Beschleunigung. Σ_2 sei ruhend; es befinde sich aber in einem homogenen Gravitationsfelde, das allen

> Gegenständen die Beschleunigung $-\gamma$ in Richtung der X-Achse erteilt. Soweit wir wissen, unterscheiden sich die physikalischen Gesetze in bezug auf Σ_1 nicht von denjenigen in bezug auf Σ_2; es liegt dies daran, daß alle Körper im Gravitationsfelde gleich beschleunigt werden. Wir haben daher bei dem gegenwärtigen Stande unserer Erfahrung keinen Anlaß zu der Annahme, daß sich die Systeme Σ_1 und Σ_2 in irgendeiner Beziehung voneinander unterscheiden, und wollen daher [...] die völlige physikalische Gleichwertigkeit von Gravitationsfeld und entsprechender Beschleunigung des Bezugssystems annehmen." [p. 454 of 12]

A consequence is that a local measurement in a system freely falling in some gravitational field no gravitational field is measured. With this in mind it is possible to see why defining mass is an issue. By the famous $E = mc^2$ mass and energy are equivalent. It is a basic fact of general relativity that the gravitational field has itself an energy or, equivalently, a mass. Thus, the mass of a system consists of two components, the mass of the matter, which creates the gravitational field, and the energy/mass "stored" in the gravitational field. To quantify the full mass one would like to add the two parts. However, as just discussed, a laboratory freely falling does not measure any gravitational field and this is the case for any local measurement at any point of the gravitational field. Therefore, it is a direct consequence of the equivalence principle that it is not possible to measure the mass of the gravitational field locally. But then half of the components which make up the mass of a system are missing. It is due to this consequence of the equivalence principle and the equivalence of mass and energy that a definition of mass is highly non-trivial in general relativity. That the space(time) on which the systems "lives" is altered by the system is what fundamentally distinguishes general relativity from other theories and the reason why there are no problems with defining mass in other theories. At this point it might seem that instead of asking why there is a problem with defining mass one now should question whether the mass of a system can be defined at all. It turns out that this is indeed possible and the precise way this is done was the breakthrough in the early 1960s alluded to above. We will review the definition of mass (in four dimensions) in chapter 4. For now, note that we always wrote that a *local* measurement cannot measure the gravitational field and indeed the idea is to not do local but global measurements. By going far away from the system and then performing the measurement it is possible to account for both components of the mass in a suitable manner and the mass of a system can be defined.

After the question whether it is possible to define the mass of a system was answered in the affirmative a new problem came up, namely, is the mass positive? As before, the question seems strange at first and a justification of why this is a problem is in order. The ground state of a spacetime in general relativity is set to be the flat space to which one assigns zero energy. Thus, the point of zero energy is fixed by the situation that there is no matter and no gravitational field. In Newtonian gravity a bound system has negative energy but in general relativity this would be a problem since negative energy is equivalent to negative mass which would lead to a repelling rather than attracting gravitational force. Additionally, there might be gravitational radiation which reduces the mass of a system by carrying energy away. Together with the indirect way of defining the mass described above this leads to the question whether mass is positive and Minkowski space a stable ground state. If the mass were not positive this would signify an inherent instability in the theory. It has turned out to be remarkably difficult to establish this result and a proof for rather general conditions was found only in 1979 by R. Schoen and S. Yau [13] and, in 1981, E. Witten found a significantly simpler proof exploiting spinor techniques [14]. These proofs settled the debate about whether mass is positive in four dimensional general relativity. We will discuss this further in chapter 4.

The goal of this thesis is to find an expression of the so-called Bondi mass (one of the masses that can be defined), in higher odd dimensions $d \geq 5$ and to show that it is positive. The question of whether aspects of four dimensional gravity carry over to higher dimensions is a delicate issue and, a priori, very unclear. In some cases higher dimensions behave just like the usual four dimensional theory, in other cases there are differences. Additionally, and perhaps surprisingly, there are crucial differences between even and odd dimensions. For example, in four dimensions there is a so-called memory effect basically saying that a system of test particles, which are exposed to gravitational radiation, may be permanently displaced by the radiation such that there remains a memory of the perturbation in the system. This effect has been known for more than 40 years and is related to asymptotic symmetries. However, it turns out that the memory effect does not exist in higher dimensions [15, 16]. Another example is the definition of conformal null infinity. Here, the definition from four dimensions can be adopted to even higher dimensions but not to odd ones [17, 18]. These two examples illustrate that the role of dimensions in general relativity is a non-trivial and interesting issue. It turns out that the idea of the proof of positivity carries over to higher dimensions but the proof is considerably more difficult. This is because the physically relevant terms are not of sub-leading order in an asymptotic series expansion in distance r but fall off much slower as $r \to \infty$. In particular, the deviation from Minkowski space due to radiation falls of slower than the deviation which is present near spatial infinity while in four dimensions they fall-off is of the same order. An expression for the Bondi mass and its positivity has been found in even dimensions [17, 19]. In chapter 6 we derive a coordinate expression for the Bondi mass in odd dimensions and chapter 7 contains the proof of the positivity of the Bondi mass in odd dimensions. These are the two main results of this thesis.

A more detailed outline is as follows. In the first part we give background information and introduce some rather well-known concepts. For the proof of positivity we need spinors on curved spacetime. They are motivated and introduced in chapter 3. In chapter 4 we look primarily at gravity in four dimensions. In section 4.1 we introduce Bondi coordinates in the conformal framework and it is discussed why we have to use Bondi coordinates in odd higher dimensions. Section 4.2 contains a review of the classical results concerning Bondi and ADM mass and we derive some of the expressions. The question of positivity is considered in more detail in section 4.3 where we sketch the proof due to Witten. In the second part we first look at gravitational waves as an example of general relativity in odd dimensions in section 5.1 and we then summarize our assumptions and specify the spacetime we will work on for the remainder of the thesis in section 5.2. Afterwards, we start considering the Bondi mass in higher dimensions. We then 6.1, investigate how the vacuum Einstein equations can be used to investigate the metric in Bondi coordinates and find some structure. The results of this investigation are used in section 6.2 to derive a coordinate expression for the Bondi mass in odd dimensions. This is the first main result of this thesis. Chapter 7 contains the proof of positivity of the Bondi mass in odd dimensions, the most important result of this work. In section 7.1 we adapt the general definitions of spinors and gamma matrices to the manifold and coordinates we chose. A outline of the proof is given in section 7.2. Then, the proof follows in the subsequent sections 7.3-7.9. In chapter 8 we discuss our results and compare them with results in four dimensions and works of other authors in higher dimensions. We summarize the thesis and give an outlook in chapter 9. The third part contains some appendices. In appendix A we write down the components of the Ricci tensor in Bondi coordinates and appendix B contains an auxiliary computation relating the spin derivative in the physical and unphysical spacetime. A brief, non-rigorous summary of how to define conserved quantities in general relativity and how to derive a geometric expression for the Bondi mass can be found in appendix C. Finally, appendix D explains the idea of holonomy and it is outlined how this can be used to classify manifolds which admit Killing spinors.

CHAPTER 2

Notation and Glossary

Some important notations and the chapter/section where they are introduced are listed in the following. Some symbols are also used with a different definition in some places if the meaning is clear from context. (M, g) always denotes a Riemannian or Lorentzian manifold. Throughout the thesis we use geometric units $G = c = 1$.

$\epsilon_{a_1 a_2 \ldots}$ (with indices)	Volume element	
TX	Tangent space of X	
∇	Levi-Civita derivative of g	
$\star$	Hodge star operator	
Γ^a_{bc}	Christoffel symbol	
R_{abcd}, R_{ab}, R	Riemann-, Ricci tensor, Ricci scalar	
$\Re(x), \Im(x)$	Real part, imaginary part of x	
$x_{[a}y_{b]}$	$\equiv \frac{1}{2}(x_a y_b - y_a x_b)$ Antisymmetrization	
$x_{(a}y_{b)}$	$\equiv \frac{1}{2}(x_a y_b + y_a x_b)$ Symmetrization	
$Cl(V, q), Cl_{r,s}, Cl^c_{r,s}$	Clifford algebras	Section 3.3.1
dot "$\cdot$"	Clifford Multiplication (for spinors)	Section 3.3.1
$\mathscr{S}(\mathscr{M})$	(Complex) spinor bundle	Section 3.3.3, Section 7.1.1
ϵ (without indices)	Killing spinor	Section 3.4
(u, r, x^A)	Bondi coordinates	Section 4.1.2
γ_{AB}	"Round" metric	Section 4.1.2
$\mathscr{I}^+$	Conformal null infinity	Section 4.1
$(\tilde{M}, \tilde{g})$	Unphysical spacetime	Section 4.1.1
N_{AB}	News tensor	Section 4.2.2, Section 6.2
$(\mathscr{M}, g)$	odd-d Lorentzian, spin manifold	Section 5.2
$x^{(n)}$	nth coefficient of series expansion in r	Section 5.2
$\dot{x}$	r-derivative of x	Section 5.2
x'	u-derivative of x	Section 5.2
(Σ, s)	$(d-2)$-dimensional spin manifold	Section 5.2
$\mathcal{D}, D$	Levi-Civita/spin derivative of s_{AB}, γ_{AB}	Section 5.2
∇	Spin derivative of g	Section 7.1.5
$\mu_{\tilde{g}}$	Bondi mass density	Section 6.2
C_g, K_g	Weyl, Schouten tensor of g	Section 6.2

m_Σ	Bondi mass of Σ	Section 6.2
$Q(X,Y)$	(Witten-Nester) 2-form:	Section 7.1.1
e_a^μ, $\tilde{e}_a^\mu$	Tetrad	Section 7.1.2
σ^a, $\tilde{\sigma}^a$	Gamma matrices in curved spacetime	Section 7.1.3
$P_\pm$	Projectors	Section 7.1.4
Γ_A (one index)	Gamma matrix on subspace	Section 7.1.4
$\mathcal{H}$	Hypersurface	Section 7.1.6

Part I

Fundamentals

CHAPTER 3

Spinors

This chapter starts with a very brief, non-rigorous discussion of the notion of principal G-bundle and associated bundle in section 3.1. The main purpose is not to give a pedagogical introduction but to fix notations. We also discuss, as an example, how the tangent bundle of a manifold can be defined as an associated bundle. This simple example already shows all ingredients necessary to define spinors such that it can be used as a well-known reference for the construction of spin bundles. In section 3.2 we motivate why spinors are relevant and how they are related to group theory by looking at the relation of $SU(2)$ and $SO(3)$. Thereafter, in section 3.3, we introduce Clifford algebras which are then used to define spin groups and the relevant representations. This section is rather algebraic and relatively unrelated to the previous sections. However, once we have defined the spin group we can make the connection with geometry by going back to the topic of section 3.1 and defining spin bundles and spinors as associated bundles and sections thereof. Finally, we define the notion of Killing spinor fields in section 3.4. This concept will be crucial in the proof of mass positivity.

3.1 Principal G-bundle and Associated Bundle

We recall the definition of the principal and associated bundle and fix some notations we will need; we refer to the literature, e.g. [20–23], for an introduction. A **principal G-bundle**, where G is a topological group, is a fiber bundle $\pi : U \to M$ together with a right action $U \times G \to U$ such that

$$y \in U_x \Rightarrow yg \in U_x \quad \forall g \in G \tag{3.1}$$

and G acts transitively (thus the orbits of the G-action are the fibers) and freely on the fibers. Thus, each fiber of the principal G-bundle is homeomorphic to G and locally the bundle is equal to $T \times G$ where T is an open subset of M, see Figure 3.1. The orbit space U/G is homeomorphic to M.

Given a principal G-bundle U, there is an associated bundle, which is constructed as follows. Let F be another space. An action of G on $U \times F$ is given by $(u, f) \to (ug, g^{-1}f)$ where $g \in G$, $u \in U$, and $f \in F$. The **associated bundle** $p : E \to M$ is defined to be the quotient space (space of orbits) $E := U \times F/G$ where (u, f) and $(ug, g^{-1}f)$ are identified. This defines an equivalence class $[u, f] = \{(u, f) \cdot g \equiv (ug, g^{-1}f) : g \in G\}$. In the special case that F is a vector space V, let

ρ be a representation of G on V. The associated bundle is written $U \times_\rho V$. The two points (u, v) and $(ug, \rho(g)^{-1}v)$ of $U \times V$, $v \in V$, are identified in $U \times_\rho V$.

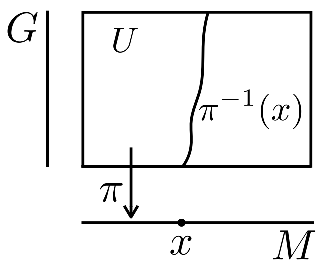


Figure 3.1: Principal Bundle

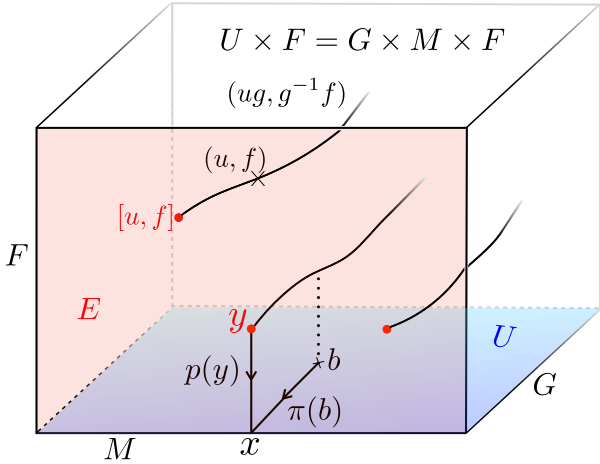


Figure 3.2: Associated Bundle

3.1.1 Example: Frame Bundle and Tangent Bundle

As an example we want to construct a principal G-bundle, which gives us bases of the tangent space at each point (the frame bundle), and, as an associated bundle, the tangent bundle. We start by looking for a principal bundle whose fiber above any point $x \in M$ consists of all ordered bases for the tangent space T_xM at x. As a group we thus choose the general linear group $Gl(n, \mathbb{R})$ which, with its natural action, is known to carry one basis into another. If there is additional structure on M (e.g. a Riemannian metric) then certain bases are distinguished (e.g. orthonormal bases) that reduce the structure group from $Gl(n, \mathbb{R})$ to some subgroup (e.g. $SO(n)$). We will not assume any additional structure, but the following discussion can easily be modified to accommodate this case. A **frame** at $x \in M$ is an ordered basis $B = (b_1, ..., b_n)$ for T_xM such that there is a natural isomorphism between the standard basis $\{e_i\}$ of $\mathbb{R}^n$ and B given by $e_i \mapsto b_i$. Then, we define $P_{Gl}(M)_x$ as the set of all frames at x and $P_{Gl}(M) := \cup_{x \in M} P_{Gl}(M)_x$. There is a surjective map $\pi : P_{Gl}(M) \to M$ defined fiberwise by $\pi(B) = x$. Together with the natural action $\alpha : P_{Gl}(M) \times Gl(n, \mathbb{R}) \to P_{Gl}(M)$, $\alpha(p, g) \mapsto p \cdot g$ of $Gl(n, \mathbb{R})$ on $P_{Gl}(M)$ we have the principal $Gl(n, \mathbb{R})$-bundle over M, $\pi : P_{Gl}(M) \to M$, called **frame bundle** over M. A local cross section $s : U \subseteq M \to P_{Gl}(M)$ assigns to each $x \in U$ a frame $s(x) = (b_1(x), ..., b_n(x))$ defining the **frame field**.

It is natural to look at the vector bundle (that is, the fiber is a vector space V) associated with the frame bundle and given by the representations of $Gl(n, \mathbb{R})$ on the vector space. We consider the natural (matrix) representation $\rho : Gl(n,\mathbb{R}) \to Gl(\mathbb{R}^n)$ of $Gl(n,\mathbb{R})$ on the vector space $\mathbb{R}^n$. For $g \in Gl(n,\mathbb{R})$ and $v \in \mathbb{R}^n$ the representation is given by $\rho(g)v = g \cdot v = gv$ with the usual matrix multiplication in the last term. We have now all parts needed to define an associated vector bundle. Associated to the frame bundle $P_{Gl}(M)$ is a bundle

$$TM := P_{Gl}(M) \times_\rho \mathbb{R}^n \,, \tag{3.2}$$

the **tangent bundle** of M. The fibers of this bundle are just the usual tangent spaces T_xM at each point $x \in M$. (Note, that other choices of the representation are possible and yield other bundles, e.g. choosing $\rho(g) = (g^T)^{-1}$ yields the cotangent bundle.) A cross section of the tangent bundle yields a **vector field** on M. In this context it is very easy to make sense of the abstract definitions. The associated bundle consists of points obtained by defining the equivalence class $[B,v] = \{(p,v)\cdot g \equiv (pg, g^{-1}v) : g \in Gl(n,\mathbb{R})\}$. We have the frame $B = (b_1, \dots, b_n)$ and in these coordinates the vector can be written as $v = v^i b_i$. Then, the part $pg = (\tilde{b}_1, \dots, \tilde{b}_n)$ simply is the basis in new coordinates given by $\tilde{b}_j = b_i g^i_j$ while the part $g^{-1}v = (\tilde{v}_1, \dots, \tilde{v}_n)^T$, $\tilde{v}^j = g^j_i v^i$, transforms the coordinates of the vector, seen as a n-tuple, such that in the end the vector does not change,

$$\tilde{v}^j \tilde{b}_j = g^j_k v^k b_l g^l_j = v^i b_i \,. \tag{3.3}$$

Thus, the set $\{(pg, g^{-1}v) : g \in Gl(n,\mathbb{R})\}$ simply contains all possible descriptions (in all bases available) of a given tangent vector v at x and $[B,v]$ is the vector independent of the basis chosen. Hence, this construction of the tangent bundle as an associated bundle of the frame bundle is just an elaborate way of saying that a vector is an object transforming "in the right way" under a change of coordinates. While this "coordinate dependent" definition might seem old-fashioned it turns out that spinors have to be defined very similar, i.e., as objects which transform under a given group "in the right way".

The remainder of this chapter aims specifying what group has to be taken to replace $Gl(n,\mathbb{R})$ and what "in the right way" actually means for spinors. In section 3.2 we heuristically motivate the rather abstract definitions to be discussed in subsequent sections by looking at the example of $SU(2)$ and $SO(3)$. Then, in section 3.3, we define the Clifford algebra and use this to define the spin group (which will replace $Gl(n,\mathbb{R})$) that is used to construct the associated bundle whose cross sections will yield spinor fields.

3.2 Motivation

The constructions and definitions in the next sections are rather abstract. This section will provide a motivation and show the basic idea behind the constructions in the following sections by looking at a concrete example, namely the groups $SU(2)$ and $SO(3)$, and how these are related to the definition of spinors as well as the general idea behind the construction of spinor fields on curved spacetimes. For a more detailed and explicit discussion of this example see e.g. [22, 24].

We start with some facts about the topology of $SU(2)$ and $SO(3)$. $SU(2)$ is simply connected, which can be seen visually by noting that $SU(2)$ is isomorphic to a 3-sphere that is simply connected. In contrast, $SO(3)$ is connected, but not simply connected; the fundamental group is $\pi_1(SO(3)) = \mathbb{Z}_2$. $SO(3)$ is isomorphic to the projective space $P\mathbb{R}^3$, which is geometrically a 3-sphere with antipodal points identified. Due to the identification of antipodal points, not all curves on the sphere are homotopic to the identity. We are now interested in $SO(3)$ as the group of rotations on $\mathbb{R}^3$, i.e., the matrix representation of $SO(3)$ on $\mathbb{R}^3$. From this point of view the fact that $SO(3)$ is not simply connected has the following consequence. Consider the curve in $SO(3)$ which is a family of rotations about a fixed axis traversed once from 0 to 2π. This curve cannot be deformed continuously to the identity, since it is a loop in $SO(3)$ and thus not homotopic to the identity. However, the same curve traversed twice is homotopic to the identity. To see the relevance of this a look at the relation between $SO(3)$ and $SU(2)$ is useful. There exists a homomorphism, called Ad, from $SU(2)$ onto $SO(3)$ and the two groups are locally the same (isomorphic Lie algebras, $\mathfrak{su}(2) \cong \mathfrak{so}(3)$), i.e., when looking only at infinitesimal rotations the action of both groups is indistinguishable. The map is defined as

$$\mathrm{Ad} : SU(2) \to SO(3), \quad \mathrm{Ad}(g)Y = gYg^{-1} \tag{3.4}$$

where $g \in SU(2)$ and $Y \in \mathfrak{su}(2)$. Because $\mathfrak{su}(2)$ is as a vector space isomorphic to $\mathbb{R}^3$, Ad defines an action of $SU(2)$ on $\mathbb{R}^3$, which in fact is equal to the matrix representation of $SO(3)$ on $\mathbb{R}^3$. Ad is onto and thus every rotation in $\mathbb{R}^3$ (that is, every element of $SO(3)$) has a corresponding element in $SU(2)$. However, the elements of both groups are not in a one-to-one correspondence. Since the map Ad is $2:1$ ($\ker(\mathrm{Ad}) = \mathbb{Z}_2$) there are two elements $u \in SU(2)$ for each $R \in SO(3)$; $\mathrm{Ad}(\pm g) = R \in SO(3)$. This can be seen from (3.4) where, due to the conjugation, the minus signs cancel. Given an axis of rotation $\mathbf{n}$ and an angle θ, let $\boldsymbol{\sigma}$ be the vector of the three Pauli matrices (which are a basis of $\mathfrak{su}(2)$) and let $\mathbf{E}$ be the vector of the basis of $\mathfrak{so}(3)$. Then,

$$\mathrm{Ad}(u) \equiv \mathrm{Ad}\exp\left(\frac{\boldsymbol{\sigma}}{2i} \cdot \mathbf{n}\theta\right) = \exp\left(\mathbf{E} \cdot \mathbf{n}\theta\right) \equiv R \tag{3.5}$$

The factor of $1/2$ in u is responsible for the $2:1$-mapping, since, as one easily sees, rotations with $\theta = 2\pi$ and $\theta = 4\pi$ yield different values for $u \in SU(2)$, namely -1 and $+1$, while both give the same value for $R \in SO(3)$. Therefore, one is lead to study the rotations/representations of $SU(2)$ instead of $SO(3)$, since, using Ad, one can always return to $SO(3)$ and $SU(2)$ encompasses a finer/more interesting structure. As we have seen, $SU(2)$ acts on $\mathbb{R}^3$ via the adjoint representation.

but its fundamental representation is on $\mathbb{C}^2$. Now, let $g \in SU(2)$ induce a change of basis $\mathbb{R}^3 \to \mathbb{R}^3$ (via the adjoint representation). The same element $g \in SU(2)$ can act on $\mathbb{C}^2$ (via the fundamental representation) and thus there is a transformation of the element $(\psi_1, \psi_2) \in \mathbb{C}^2$ associated to the change of basis $\mathbb{R}^3 \to \mathbb{R}^3$. This element is called a spinor. Thus, $SU(2)$ is also called Spin(3) in this context. In higher dimensions, Spin(n) denotes the universal cover of $SO(n)$, although Spin(n) is not generally given by $SU(n)$. A very similar construction relates the identity component of the Lorentz group and its double cover $Sl(2, \mathbb{C})$. Thus, spinors are defined as elements of a complex vector space that transform under the double cover of the orthogonal group changing the basis of physical space.

In curved spacetimes this entails the following. Take two observers O_1 and O_2 at a point $p \in M$ in (M, g), where each observer is represented by an orthonormal tetrad at p. There is a Lorentz transformation (acting on T_pM) rotating the tetrad of O_2 into that of O_1. In the next section, this will be done by defining an action of the orthogonal group on the tangent bundle. Consequently, spinors are defined as elements of a complex vector space which "transform in the right way". For each Lorentz transformation there is an associated spinor transformation $\psi \to \Lambda\psi = \psi'$ such that all measurements made by O_2 on ψ' yield the same results as all measurements of O_1 on ψ. In the next section this is done by lifting the orthogonal group to a spin group which then is used via a representation on some complex vector space to define spinors as elements of this space.

3.3 Spinors

The aim of this section is to introduce the notion of a spinor field on a manifold. A crucial ingredient is the Clifford algebra, a generalization of the well-known algebra generated by Pauli and gamma matrices. This topic is very broad and only a few important notions/definitions and properties are introduced in the following subsection. Afterwards, spinor groups are introduced which define spinors via a representation on an appropriate vector space. Lastly, we will briefly describe how these concepts can be used to define spinor fields and the Dirac operator on (pseudo-)Riemannian manifolds in arbitrary dimensions. More complete treatments of Clifford algebras and spinors can be found in a plethora of books and review articles, we follow mostly [23, 25–28].

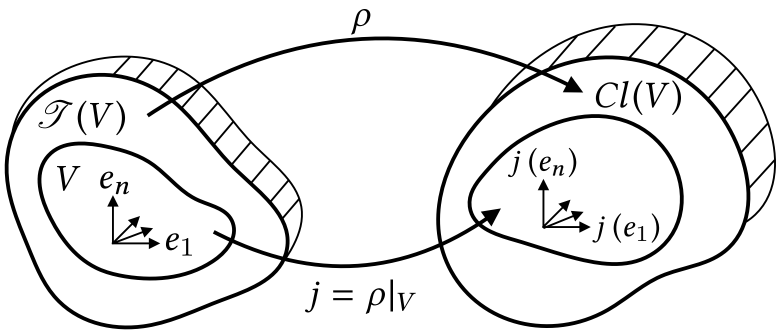


Figure 3.3

3.3.1 Clifford Algebra

Let V be a finite-dimensional vector space over a field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). Define a quadratic form q on the field. Denote the tensor algebra over V as

$$\mathscr{T}(V) := \bigoplus_{k=0}^{\infty}\left(\bigotimes^{k} V\right). \tag{3.6}$$

To proceed we need the following basic concept from ring theory. Let $(R, +)$ the additive group of a ring $(R, +, \cdot)$. If

i) $(\mathscr{I}, +)$ is a subgroup of $(R, +, \cdot)$, and

ii) $\forall x \in \mathscr{I}, \forall r \in R$: $x \cdot r, r \cdot x \in \mathscr{I}$,

$\mathscr{I}$ is called an **ideal**. Now, let $\mathscr{I}_q(V)$ be the ideal in $\mathscr{T}(V)$ generated by all elements of the form $\{v \otimes v - q(v)1\}$ where $v \in V$. Loosely speaking, $\mathscr{T}(V)$ is the most general algebra containing the vector space V (i.e. all other algebras that contain V are in $\mathscr{T}(V)$) and by taking a quotient with the ideal $\mathscr{I}_q(V)$ we can remove all elements of $\mathscr{T}(V)$ which do not fulfil the relation we want to dictate in the Clifford algebra. Thus, the **Clifford algebra** $Cl(V, q)$ of the quadratic space (V, q) is defined to be the quotient

$$Cl(V, q) := \mathscr{T}(V)/\mathscr{I}_q(V). \tag{3.7}$$

For a given pair (V, q) the Clifford algebra is unique up to isomorphism [Corollary 5.1.3 in 23]. If the $\mathbb{K}$-vector space (V, q) is n-dimensional the corresponding Clifford algebra has dimension 2^n [Corollary 5.1.8 in 23]. The projection $\rho : \mathscr{T}(V) \to Cl(V, q)$ is an algebra homomorphism and the restriction of ρ to V yields a linear mapping $j := \rho|_V : V \to Cl(V, q)$. This map is injective and fulfils

$$j(v)^2 = q(v)1 \quad \forall v \in V. \tag{3.8}$$

Defining the symmetric bilinear form

$$2\eta(u, v) = q(u + v) - q(u) - q(v), \quad u, v \in V, \tag{3.9}$$

we have

$$j(u) \cdot j(v) + j(v) \cdot j(u) = 2\eta(u, v). \tag{3.10}$$

Thus, any basis $\{\mathbf{e}_1, ...\mathbf{e}_n\}$ of V generates, together with the identity 1, the algebra $Cl(V, q)$ multiplicatively. That is, the basis elements fulfil the relations $j(\mathbf{e}_k) \cdot j(\mathbf{e}_l) + j(\mathbf{e}_l) \cdot j(\mathbf{e}_k) = 2\eta(\mathbf{e}_k, \mathbf{e}_l)$, which defines the Clifford algebra, and the 2^n elements

$$1, j(\mathbf{e}_{i_1}) \cdot ... \cdot j(\mathbf{e}_{i_k}), \quad 1 \leq i_1 < ...i_k \leq n,\ 1 \leq k \leq n \tag{3.11}$$

are a vector space basis of the Clifford algebra [1], see Fig. 3.3.

There is a map $p \in \mathrm{Aut}(Cl(V, q)), p \circ j(v) = -j(v)$ called **parity automorphism** of $Cl(V, q)$. It induces a $\mathbb{Z}_2$-grading of the Clifford algebra,

$$Cl(V, q) = Cl^0(V, q) \oplus Cl^1(V, q) \tag{3.12}$$

where $Cl^i(V, q) = \left\{a \in Cl(V, q) : p(a) = (-1)^i a\right\}$. Additionally, there is map on $\mathscr{T}(V)$ given by $v_1 \otimes ... \otimes v_r \to v_r \otimes ... \otimes v_1$ preserving the ideal. Thus, there is an anti-automorphism $(\ )^t : Cl(V, q) \to Cl(V, q)$, $(a \cdot b)^t = b^t \cdot a^t$ called **transpose**. The two maps p and $(\ )^t$ commute.

Given a Clifford algebra over a real vector space, there is an associated Clifford algebra, which is obtained by complexification. More precisely, let (V, q) be a real quadratic space and let $(V_{\mathbb{C}}, q_{\mathbb{C}}) \equiv (V \otimes \mathbb{C}, q \otimes \mathbb{C})$ be its complexification. Then,

$$Cl(V_{\mathbb{C}}, q_{\mathbb{C}}) \cong Cl(V, q) \otimes_{\mathbb{R}} \mathbb{C} \tag{3.13}$$

is an algebra isomorphism [Proposition 5.1.14 in 23]. The Clifford algebra $Cl(\mathbb{R}^{r+s}, q)$ with $q(\mathbf{x}) = -x_1^2 - ... - x_r^2 + x_{r+1}^2 + ... + x_{r+s}^2$, where $\mathbf{x} = (x_1, ..., x_{r+s})$ is the standard basis of $\mathbb{R}^{r+s}$, will be denoted $Cl_{r,s}$ and its complexification will be denoted $Cl^c_{r,s}$. If one index is zero it is not written, i.e. $Cl_{0,n} \equiv Cl_n$. In the following, let $r + s = n$.

As is often the case in physics, the Clifford algebra enters via a representation on an appropriate vector space. By representation of a Clifford algebra we mean the following. Let (V, q) be a quadratic vector space over a commutative field $k \subset \mathbb{K}$. Then a **$\mathbb{K}$-representation** of the Clifford algebra $Cl(V, q)$ is a k-algebra homomorphism

$$\rho : Cl(V, q) \to \mathrm{Aut}_{\mathbb{K}}(W). \tag{3.14}$$

[1] As vector spaces, $Cl(V, q)$ and the exterior algebra ΛV are isomorphic. [Corollary 5.1.10 in 23]. For $q = 0$ they are isomorphic as algebras.

where W is a finite-dimensional vector space over $\mathbb{K}$ and called $Cl(V,q)$-module. To simplify notation we introduce the **Clifford multiplication**

$$\varphi \cdot w := \rho(\varphi)(w)\,, \tag{3.15}$$

where $\varphi \in Cl(V,q)$ and $w \in W$.

For Cl_n^c one finds [Proposition 5.1.19 in 23]

$$Cl_{2k}^c \cong \mathbb{C}(2^k), \quad Cl_{2k+1}^c \cong \mathbb{C}(2^k) \oplus \mathbb{C}(2^k). \tag{3.16}$$

which have the (irreducible) representations

$$\gamma_{2k} : Cl_{2k}^c \to \mathrm{End}(\Delta_{2k}) \tag{3.17}$$

and

$$\gamma_{2k+1} : Cl_{2k+1}^c \to \mathrm{End}(\Delta_{2k+1}) \oplus \mathrm{End}(\Delta_{2k+1}) \tag{3.18}$$

The space

$$\Delta_n := \mathbb{C}^{2^f}, \quad f = \left\lfloor \frac{n}{2} \right\rfloor \tag{3.19}$$

is called the **complex n-spinor module** and the representation γ_n is called the **spin representation** of Cl_n^c. We will often write γ instead of γ_n. On Δ_n there exists a positive-definite hermitian inner product $\langle\, ,\, \rangle$.

For example, the Clifford algebra appears in physics when introducing gamma matrices, for example to write down the Dirac equation. To see this note that the Clifford algebra of Minkowski space (M,η) is $Cl_{1,3}$. One can show [Table II in 26] that $Cl_4^c = Cl_{1,3} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}(4)$ where $\mathbb{C}(4)$ denotes the algebra of 4×4 complex matrices. Thus, the generators of the complexification of the Clifford algebra of Minkowski space can be represented by such matrices. A standard choice is

$$\gamma : M \subseteq Cl_4^c \to \mathbb{C}(4) \subseteq \mathrm{End}(\mathbb{C}^4), \quad \gamma(\mathbf{e}_\mu) = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix} \tag{3.20}$$

where $\mathbf{e}_\mu$ is the standard basis of $\mathbb{R}^4$, σ_μ are the Pauli matrices[2], and $\tilde{\sigma}_0 = \sigma_0$ while $\tilde{\sigma}_i = -\sigma_i$. Let $\gamma_\mu \equiv \gamma(\mathbf{e}_\mu)$. Then

$$\gamma_\mu \cdot \gamma_\nu + \gamma_\nu \cdot \gamma_\mu = 2\eta_{\mu\nu} 1 \tag{3.21}$$

by virtue of the commutation relations of the Pauli matrices. Therefore, γ extends to an algebra isomorphism $Cl_4^c \to \mathrm{End}(\Delta_4)$.

[2]The Pauli matrices generate the Clifford algebra of $\mathbb{R}^3$.

3.3.2 Spinor Groups

The Clifford algebra can be used to define Spin groups that are the double cover of the orthogonal group of a given quadratic form. This will be done in this section.

Elements $v \in V$ with $q(v) \neq 0$ have an inverse $v^{-1} = v/q(v)$ and the subset $Cl(V,q)^*$ of $Cl(V,q)$ containing all invertible elements is a Lie group of dimension 2^n. The **Clifford group** of (V,q) is a Lie subgroup defined as

$$\Gamma(V,q) := \left\{a \in Cl(V,q)^* : p(a)va^{-1} \in V \; \forall v \in V\right\} \tag{3.22}$$

and it has a natural representation

$$\tilde{\mathrm{Ad}} : \Gamma(V,q) \to \mathrm{Aut}(V), \quad \tilde{\mathrm{Ad}}(a)v := p(a)va^{-1} \tag{3.23}$$

called the twisted adjoint representation. Let $O(V,q)$ be the orthogonal group of (V,q), the subgroup of $\mathrm{Aut}(V,q)$ leaving q invariant and let $SO(V,q)$ be the subgroup of elements with determinant 1. Define the norm mapping $N : Cl(V,q) \to Cl(V,q)$, $N(a) := a\,p(a^t)$. Note that for $v \in V$ this reduces to $N(v) = -q(v)$. The so-called **pin group** is defined as

$$\mathrm{Pin}(V,q) := \{a \in \Gamma(V,q) : N(a) = 1\}, \tag{3.24}$$

and the **spin group** as

$$\mathrm{Spin}(V,q) := \mathrm{Pin}(V,q) \cap \left(\Gamma(V,q) \cap Cl^0(V,q)^*\right). \tag{3.25}$$

Thus, for $a \in \mathrm{Spin}(V,q)$, we have $N(a) = a\,p(a^t) = aa^t = 1$. Note the similarity to the well-known relation for the usual orthogonal groups. In fact, $\mathrm{Spin}(n,\mathbb{R})$ is the double covering of $SO(n,\mathbb{R})$ and for $n > 2$ it is the universal cover since $\mathrm{Spin}(n)$ is simply connected in this case. We have $\rho(a)w = awa^t$, $w \in W$, as the representation on a vector space W and this gives an explicit double covering of $SO(n)$ by $\mathrm{Spin}(n)$ since $\rho(a) = \rho(\pm a)$. The spin group and special orthogonal group corresponding to $Cl_{r,s}$ are denoted by $\mathrm{Spin}_{r,s}$ and $SO_{r,s}$, respectively. Both are Lie groups. $\mathrm{Spin}_{r,s}$ is a double covering of the identity component $SO^0_{r,s}$. The covering is universal if $r > 2, s = 0,1$ or $s > 2, r = 0,1$. We will always assume that this is the case in the following. The relations between pin group and orthogonal group are similar.
Well-known examples that appear in physics are $\mathrm{Spin}(2) \cong U(1)$, $\mathrm{Spin}(3) \cong SU(2)$, and $\mathrm{Spin}_{1,3} \cong Sl(2,\mathbb{C})$. The last one, the double cover of the Lorentz group, is needed to define spinors in four dimensional spacetime. Since $\mathrm{Pin}_{r,s}$ and $\mathrm{Spin}_{r,s}$ are in Cl^c_n (with $r+s=n$) the representation of Cl^c_n restricts to faithful representations of these groups. This representation is called **spinor representation**, it is unitary with respect to the inner product defined above [Satz on p. 26 in 29]. The complex spin group $\mathrm{Spin}^c_{r,s}$ is the subgroup of $Cl^c_{r,s} \cong Cl_{r,s} \otimes \mathbb{C}$ generated by

$\text{Spin}_{r,s} \subset Cl_{r,s}$ and $U(1) \subset \mathbb{C}$. Again, the discussion above is the same for the complexifications with the obvious replacements. An explicit, less abstract discussion of all concepts introduced so far can be found in [Chapter 19 of 22] in the context of the Dirac operator on Minkowski space. A discussion of the following section in the case of four dimensions can also be found there.

3.3.3 Spin Geometry

Now we can define spin structures and spinors for pseudo-Riemannian manifolds (M, g). It turns out that Clifford algebras appear very naturally once a metric on the manifold is defined. The reason is the following. Let $\pi : E \to M$ be a pseudo-Riemannian vector bundle. In each fiber, $\pi^{-1}(p) = E_p$ there is a quadratic form $\langle v, v\rangle$ to be used for the construction of a Clifford algebra $Cl(E_p)$. Doing so at each point results is a bundle $Cl(E) = \cup_{p\in M} Cl(E_p) \to M$ of Clifford algebras over M called Clifford bundle of E. Using an irreducible representation of the spin group, one can then define spinor fields and the Dirac operator.

The first step is to define a spin structure on the tangent bundle of a manifold M. Let $\pi : E \to M$ be a real orientable n-dimensional pseudo-Riemannian vector bundle with $n > 2$. Choose an orientation of E and let $P_{SO}(E)$ be the bundle of oriented orthonormal frames. Let λ be the covering homomorphism $\lambda : \text{Spin}_{r,s} \to SO_{r,s}$ which has kernel $\mathbb{Z}_2$. A **spin structure** on E is a pair $(P_{\text{Spin}}(E), \Lambda)$, where $P_{\text{Spin}}(E)$ is a principal $\text{Spin}_{r,s}$-bundle over M and the bundle map Λ is a 2-sheeted covering $\Lambda : P_{\text{Spin}}(E) \to P_{SO}(E)$ such that $\Lambda(pg) = \Lambda(p)\lambda(g)$ for all $p \in P_{\text{Spin}}(E)$ and all $g \in \text{Spin}_{r,s}$. Thus, there is a diagram

$$\begin{array}{ccc} \text{Spin} & \xrightarrow{\lambda} & SO \\ \downarrow & & \downarrow \\ P_{\text{Spin}}(E) & \xrightarrow{\Lambda} & P_{SO}(E) \end{array}$$

A **spin manifold** is an oriented pseudo-Riemannian manifold with spin structure on its tangent bundle, $E = TM$. A Spin^c-manifold is defined with Spin^c instead of Spin. It turns out that a spin structure on the tangent bundle does not always exist and the existence is related to the second Stiefel-Whitney class. To define this class some standard tools from algebraic geometry need to be introduced. We do not want to do this here and refer to the literature on algebraic geometry and obstruction theory, see e.g. [20–23, 25] and for the case of four dimensional spacetime in general relativity additionally [30]. We now assume that a spin structure exists. That is, given the bundle $P_{SO}(TM) \equiv P_{SO}(M)$ of oriented orthonormal frames of the vector space TM we assume that we can lift the structure group $SO_{r,s}$ of $P_{\text{SO}}(M)$ to $\text{Spin}_{r,s}$ such that we have a bundle of frames $P_{\mathrm{Spin}}(M)$ consisting of all bases of TM that transform under the action of $\mathrm{Spin}_{r,s}$ on M. Locally, $P_{\mathrm{Spin}}(M) = U \times \mathrm{Spin}_{r,s}$ for an open set $U \subset M$.

This allows describing how spinors are related to this. As mentioned, the (irreducible) representations of the Spin group are the crucial component. To see why, note that the fundamental representation of $SO_{r,s}$ on $(\mathbb{R}^n, q)$ induces an action on the tensor algebra over $\mathbb{R}^n$ which leaves $\mathscr{I}_q(\mathbb{R}^n)$ invariant. Thus, there is a representation ρ of $SO_{r,s}$ on the Clifford algebra given by

$$\rho : SO_{r,s} \to \mathrm{Aut}(Cl_{r,s}). \tag{3.26}$$

As described in section 3.1, a representation of a group can be used to construct a new bundle associated to the principal bundle.

Let E be an oriented pseudo-Riemannian vector bundle of rank n and let $P_{SO}(E)$ be the bundle of oriented orthonormal frames. The associated algebra bundle

$$Cl(E) \coloneqq P_{SO}(E) \times_\rho Cl_{r,s} \tag{3.27}$$

is the **Clifford bundle** of E. Given an oriented pseudo-Riemannian manifold (M, g), $Cl(TM)$ is called Clifford bundle of M, denoted $Cl(M)$. This can again (c.f. the example in section 3.1) be understood as saying that coordinate transformations by $SO(n)$ on frames are "compensated" by associated transformations of Cl_n such that we have, in the end, invariance under coordinate transformation. $Cl(E)$ is a bundle of Clifford algebras over M, i.e., each fiber of the bundle is a Clifford algebra. The fiberwise multiplication in $Cl(E)$ provides the space of sections of $Cl(E)$ with an algebra structure. All operations defined for Clifford algebras carry over to the Clifford bundles.

Recall that the vector space $TM \subset Cl(M)$ generates $Cl(M)$ fiberwise. The last step is to define a vector bundle with fiber Δ_n on which there is an irreducible representation of $Cl_{r,s}$ and the Spin group, respectively. The concrete definition is as follows. Let (M, g) be a pseudo-Riemannian manifold with spin structure $(P_{\mathrm{Spin}}(M), \Lambda)$. The vector bundle

$$\mathscr{S}(M) \coloneqq P_{\mathrm{Spin}}(M) \times_\gamma N\,, \tag{3.28}$$

where N is a left module for $Cl_{r,s}$ and where $\gamma : \mathrm{Spin}_{r,s} \to SO(N)$ is the spinor representation given by left multiplication of elements of $\mathrm{Spin}_{r,s} \subset Cl^0_{r,s}$, is called the (real) **spinor bundle** of (M, g) over $Cl(M)$. A section of the spinor bundle is called a spinor field. A complex spinor bundle is defined equivalently with the complexified Clifford algebra and a complex module $N = \Delta_n$. The definition of spinor bundle and spinor fields is in complete analogy to the definition of the tangent bundle and vector fields discussed in section 3.1. In the end, we simply replaced the general linear group/orthogonal group in (3.2) by the spin group and then chose an appropriate vector space such that there again is an irreducible representation for the new group. The cross section is thereby changed from defining a vector field to defining a spinor field. In general, Δ_n has structure beyond that of a complex vector space, for example, one can define a real and imaginary part of elements of Δ_n. However, spinors inherit only the structure that is preserved under the action of the spin group and, since the real and imaginary parts mix, it is not possible to define the real part of a spinor. Furthermore, note that for $X \in T_pM$ the relation

$$\gamma(X)^2 = g(X,X)1 \tag{3.29}$$

holds.

The last part of this section introduces the Dirac operator. We begin with defining a connection on the bundle $P_{\text{Spin}}(M)$. If M admits a (pseudo-)Riemannian metric, the distinguished connection on the frame bundle $P_{SO}(M)$ is just the usual Levi-Civita connection. Since we now want to work on $P_{\text{Spin}}(M)$, we need to lift the Levi-Civita connection on $P_{SO}(M)$ to a connection on $P_{\text{Spin}}(M)$, which can be done as follows. Let (M, g) be an oriented pseudo-Riemannian spin manifold. Let ω be the Levi-Civita connection of g viewed as a principal connection on $P_{SO}(M)$. Let $d\lambda : \mathfrak{spin}_{r,s} \to \mathfrak{so}_{r,s}$ be an isomorphism of Lie algebras. The unique lift $\hat{\omega} = (d\lambda)^{-1}\Lambda^*\omega$ is the connection on $P_{\text{Spin}}(M)$, called **spin connection**. Hence, the diagram

$$\begin{tikzcd}
TP_{\text{Spin}}(M) \arrow[r, "\hat{\omega}"] \arrow[d, "\Lambda'"] & \mathfrak{spin}_{r,s} \arrow[d, "d\lambda"] \\
TP_{SO}(M) \arrow[r, "\omega"] & \mathfrak{so}_{r,s}
\end{tikzcd}$$

commutes. Just as the Levi-Civita covariant derivative derived from the Levi-Civita connection acts on vectors (the sections of the tangent bundle) the spin connection can be used to define a covariant derivative, which acts on spinors (the sections of the spinor bundle). (From a physics point of view the spin connection can be seen as the gauge field of the local Lorentz group such that the physics is not changed under a Lorentz transformation.) Let ∇ be this spin derivative (we use the same notation for the Levi-Civita derivative and the spin derivative since, when acting on vectors, they are equal) associated with the metric g, i.e., the covariant derivative on $\mathscr{S}(M)$. Then we have

$$\nabla\Phi = d\Phi + \sum_{i<j} \omega_{ij} \otimes e_i e_j \cdot \Phi, \quad \Phi \in \Gamma(\mathscr{S}(M)), \tag{3.30}$$

at $p \in M$, where $\{e_j\}$ is an orthonormal basis of T_pM, and ω_{ij} are the coefficients of the spin connection form. The Dirac operator of $\mathscr{S}(M)$ is a first-order differential operator $D : \Gamma(\mathscr{S}(M)) \to \Gamma(\mathscr{S}(M))$. In coordinates,

$$D\Phi = \sum_{j=1}^{n} e_j \cdot \nabla_{e_j}\Phi. \tag{3.31}$$

For Minkowski space, the Clifford bundle is trivial, we have $\nabla_{e_j} \to \partial_j$, $\mathscr{S}(M) = \mathbb{R}^4 \times \mathbb{C}^4$, and, using the definition of the Clifford multiplication, the Dirac operator reads

$$D_{\text{Minkowski}}\Phi = \sum_{j=1}^{4} \gamma_j \partial_j \Phi = \gamma^j \partial_j \Phi = \partial\!\!\!/ \Phi\,. \tag{3.32}$$

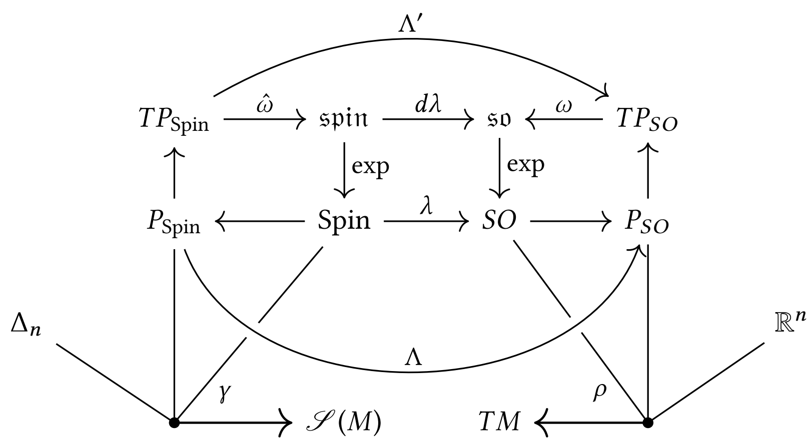
Considering $\gamma_j \equiv \gamma(e_j)$ and (3.29) (which in this case are just the commutation relations of the gamma matrices) we have

$$D^2_{\text{Minkowski}} = \gamma^i \partial_i \gamma^j \partial_j = \partial^j \partial_j, \tag{3.33}$$

the Laplacian. We summarize the results of this section in the following table comparing the results for spinors (complex case) with the well-known analogous concepts for vectors.

	Vector	Spinor
Principal bundle	$P_{SO}(M)$	$P_{\text{Spin}}(M)$
Structure group	$SO_{r,s}$	$\text{Spin}^c_{r,s}$
Representation space	$\mathbb{R}^n$	$\Delta_n := \mathbb{C}^{2^f}, \quad f = \lfloor \frac{n}{2} \rfloor$
Associated bundle	$TM = P_{SO}(M) \times_\rho \mathbb{R}^n$	$\mathscr{S}(M) = P_{\text{Spin}}(M) \times_\gamma \Delta_n$
Cross section	Vector field	Spinor field
Connection	Levi-Civita connection ω	Spin connection $\hat{\omega}$
Covariant Derivative	Levi-Civita derivative ∇	Spin derivative ∇

The numerous maps we defined are summarized in the following diagram, exp is the usual exponential map from a Lie algebra to the group.



3.4 Killing Spinor

The last important notion about spinors we will need is about Killing spinors on a Riemannian manifold. The spinor field ϵ defined on a spin manifold (Σ, s) is called a real **Killing spinor**, if there is a constant $\lambda \in \mathbb{R}\backslash\{0\}$, such that ϵ fulfils

$$\nabla_X \epsilon = i\,\frac{\lambda}{2} X \cdot \epsilon \tag{3.34}$$

for all $X \in T\Sigma$. λ is called Killing number. If $\lambda = 0$ then ϵ is called parallel spinor.

If λ is complex and non-zero then ϵ is called complex Killing spinor. Note, however, that real/complex refer only to λ, the spinor field is a section of a complex spinor bundle in either case. We will work only with real Killing spinors and refer to them simply as Killing spinor. Additionally, we now assume that the spinor field is defined on a Riemannian manifold. Some important facts about Killing spinors that are relevant to us are listed in the following, we refer to [29, 31] for proofs and further discussion. The Killing spinor ϵ is a eigenspinor of the Dirac operator with eigenvalue $-n\lambda$, where $n = \dim(\Sigma)$. A Killing spinor is a special case of a twistor spinor. Associated to the Killing vector ϵ there is a vector field

$$V_\epsilon = \sum_{i=1}^{n} (e_i \cdot \epsilon, \epsilon)\, e_i \tag{3.35}$$

which is a Killing vector field on the Riemannian manifold (Σ, s) with orthonormal basis $\{e_j\}$. This explains the name "Killing spinor". If there exists a Killing spinor on (Σ, s) then (Σ, s) is a compact Einstein manifold of positive scalar curvature

$$R = 4n(n-1)\lambda^2 > 0\,. \tag{3.36}$$

This condition is rather restrictive and it is possible to classify all manifolds, which may carry Killing spinors or parallel spinors, by their holonomy group. For parallel spinor this was done by [32, 33] and for Killing spinors by [34]. See Appendix D for a brief discussion and examples of manifolds which admit Killing spinors.

CHAPTER 4

Conformal Infinity and Mass in Four Dimensions

We begin our discussion of mass in general relativity and its positivity with a look at the four dimensional case since this is well understood and can serve as an example/motivation for the more complicated higher dimensional case. For this we first need to introduce the notion of null infinity and asymptotic flatness. We will also discuss how to construct a coordinate system, called Bondi coordinates, near null infinity. It turns out that four dimensions and even dimensions $d \geq 4$ are rather similar and thus we will discuss the more general case. However, the case of odd dimensions is different, for reasons to be explained in the first section, and thus we will discuss it in more detail only in the next chapter. The second section of this chapter consists of a review of the different definitions of mass in four-dimensional general relativity. Thereafter, we briefly discuss the problem of positivity in four dimensions.

4.1 Null Infinity and Bondi Coordinates in Even Dimensions

We start by recalling the definition of conformal null infinity in four dimensions and discuss why the definition carries over to higher even (but not odd) dimensions. Then, we construct Bondi coordinates in even dimensions following the arguments in [35–37]. Using this, we define asymptotic flatness by assuming that a suitable conformal embedding and Bondi coordinates exist.

4.1.1 Conformal Transformation

We want to find a suitable coordinate system to study the problem of an isolated system (sitting in some compact region of spacetime) which radiates gravitational waves. The challenge now is to define asymptotic flatness and "going to infinity" in a meaningful way, which is a nontrivial problem since there is no background spacetime that one could use as a reference. A possibility to deal with this is to bring the points at infinity to a finite distance thereby circumventing the problem. The procedure is similar to the compactification used to construct real/complex projective spaces, a well-known example being the Riemann sphere. We briefly and very informally describe the basic idea behind this technique which was introduced by R. Penrose in [38]. An introduction can, for example, be found in [35, 39–41]. Then, we use the introduced notions to define a suitable coordinate system near infinity. We first consider four dimensional spacetimes. As mentioned we want to transform the spacetime in such a way that "points at infinity" are brought to a finite distance. A natural requirement for a transformation of the metric is that the causal structure is not changed, that is null/timelike/spacelike vectors are mapped onto null/timelike/spacelike vectors. Otherwise, physical processes and the result of measurements would be changed by the transformation rendering a study of physical processes in the transformed spacetime worthless. Let $(\tilde{M}, \tilde{g})$ be a new spacetime, called **unphysical spacetime**, where $\tilde{M}$ is a manifold with boundary such that (M, g) can be mapped into $(\tilde{M}, \tilde{g})$ via a conformal isometry. That is, let Ω be a smooth, strictly positive function and let $\psi : M \to \psi(M) \subset \tilde{M}$ such that

$$\tilde{g} = \Omega^2 \psi^* g \,. \tag{4.1}$$

It can be shown, see e.g. [35], that the causal structure is persevered by this transformation. The transformation of the metric naturally induces transformations of all quantities which depend on the metric, e.g. the Levi-Civita derivative or the Riemann tensor. Now, making a suitable choice for Ω achieves the desired goal of bringing points infinitely far away in (M, g) to a finite distance in $(\tilde{M}, \tilde{g})$. The concrete choice of Ω depends on the situation at hand. This construction essentially adds a boundary to M and this boundary represents infinity. The boundary of the unphysical spacetime is equal to the union of null infinity (endpoints of null geodesics) and spatial infinity. We denote future null infinity be $\mathscr{I}^+$. It is important to note that this method effectively replaces the falloff conditions (in the physical spacetime) by differentiability conditions (in the unphysical spacetimes) in the definition of asymptotic flatness. Due to this the differential structure at infinity is crucial and there are some additional technical conditions about the smoothness at infinity necessary, see [35, 39–41]. If the mapping into an unphysical spacetime in the manner above is possible and if the smoothness conditions are fulfilled the spacetime is usually said to be asymptotically flat. However, we will need an additional assumption, namely the existence of a suitable coordinate system, so the existence of such a mapping is only the first assumption and does not yet define an asymptotically flat spacetime. The advantage of the conformal method is that one obtains, with weak assumptions on the differentiability of the fields, the desired definitions. The geometric methods applied do not require the introduction of a coordinate system to define infinity while simultaneously supplying a clear geometric picture of the situation at hand. Additionally, taking limits, which can be subtle, is avoided. These are reasons why the definition via conformal mappings is the preferred method in four dimensions and one would like to also use the technique in higher dimensions. Looking at the above steps one could guess that there should be no problem with the generalization to higher dimensions, since the geometric method does not seem to crucially depend on the dimension. It turns out that this guess is indeed correct for even dimensions

$\dim(M) =: d > 4$, see [17] for details, and the definitions are very similar to four dimensions. However, as shown by [18], it is not possible to define conformal null infinity for odd dimensions $d \geq 5$, as the unphysical metric is at most $(d-3)/2$ times differentiable and therefore not smooth. As mentioned above the assumptions on the differentiability of fields at infinity are crucial for the definition and hence, one is left only with the possibility of using so-called Bondi coordinates, which indeed can be generalized to arbitrary higher dimensions. We will show how Bondi coordinates are constructed in the unphysical spacetime (in even dimensions) in the next section and come back to odd dimensions in the next chapter.

4.1.2 Bondi Coordinates

Since we are not particularly interested in the source, it suffices to study the effects at large distances and the coordinates should be chosen in a way that is as simple as possible. Such a coordinate system was introduced by [11, 42, 43] and the coordinates are called **Bondi coordinates**. See also e.g. [44] for a review. The intuition behind the different coordinates is as follows. Let there be a radiating source in some (compact) region of the spacetime (M, g) and consider a family of null geodesics originating at the source at an instance of time. Similarly to spherical coordinates, there is a coordinate r quantifying the distance from the source. It is sometimes called luminosity distance and defined as the coordinate along the null geodesics. We choose an orientation such that r increases when an observer moves away from the source. Then a "point" in null infinity, the "point" to which a specific light ray travels in an infinite amount of time, is at $r \to \infty$ where the limit is taken by going along null geodesics of radiation. The second coordinate u is called retarded time (see below for the reason) and used to distinguish null geodesics corresponding to radiation that was emitted from the source at different times. u increases as time moves forward, i.e. larger u correspond to later times, see Fig. 4.1. The remaining coordinates will be denoted by $x^A := (x^1, \ldots, x^{d-2})$. These are local coordinates on the $(d-2)$-dimensional (r, u)-constant surfaces Σ.

Following [35–37] we now describe how such a coordinate system is constructed in even dimensions. It is important to note that we do not proof that such a coordinate system exists at null infinity but, in the end, impose the existence as a condition for asymptotic flatness at null infinity. We begin by constructing a coordinate system (u, Ω, x^A), where $A = 1, \ldots, d-2$, on a small open neighborhood O of an arbitrary point $p \in \mathscr{I}^+$ such that the following holds. Define

$$n^a = \tilde{g}^{ab}\tilde{\nabla}_b\Omega \tag{4.2}$$

which is null at $\mathscr{I}^+ \cap O$ and Ω is a scalar chosen to be $\Omega = 0$ on $\mathscr{I}^+ \cap O$. With a suitable gauge choice (see [35]) n^a satisfies the geodesic equation

$$n^a\tilde{\nabla}_a n^b = 0 \tag{4.3}$$

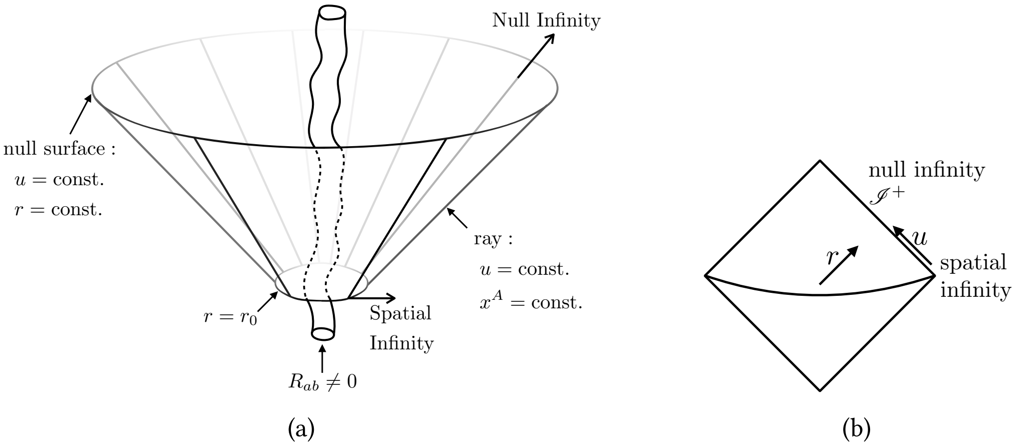


Figure 4.1: Sketches of Bondi coordinates with focus on different aspects. In (a), u is fixed and the source of matter is indicated by $R_{ab} \neq 0$. In (b), the definitions of null and spatial infinity in Bondi coordinates are sketched. Each point in this two-dimensional sketch represents a $(d-2)$-dimensional surface parametrized by x^A near infinity.

and the expansion, shear and twist of the null geodesic generators of $\mathscr{I}^+ \cap O$ vanish. Thus, n^a generates a congruence of null geodesics which do not intersect. u is defined as the affine parameter along the null geodesic generators of $\mathscr{I}^+ \cap O$ such that

$$n^a \nabla_a u = 1 \tag{4.4}$$

and thus $n^a = (\partial/\partial u)^a$. There exists a $(d-2)$-dimensional surface Σ in $\mathscr{I}^+ \cap O$ which intersects each of the null geodesic generators of $\mathscr{I}^+ \cap O$ at precisely one point. On Σ local coordinates $x^A = (x^1, ..., x^{d-2})$ can be introduced. Let $m^a_A = (\partial/\partial x^A)^a$, then we have $\tilde{g}_{ab} n^a m^b_A = 1$ and we can define a null vector field l^a on $\mathscr{I}^+ \cap O$ such that $\tilde{g}_{ab} n^a l^b = 1$ and $\tilde{g}_{ab} m^a l^b = 0$, i.e., the null geodesics generated by l^a are orthogonal to Σ and transverse to $\mathscr{I}^+ \cap O$. Let Ω denote the affine parameter on the null geodesics defined by l^a with $\Omega = 0$ on $\mathscr{I}^+ \cap O$. The x^A coordinates are required to be constant along the orbits of n^a and l^a while u is required to be constant along the curves defined by l^a. We have $n_a l^a = 1$ and $l_a m^a = 0$ everywhere on O, not just on $\mathscr{I}^+ \cap O$. In a coordinate system which fulfils these conditions the unphysical metric takes the form

$$\tilde{g} = -2\Omega^2 \alpha du^2 + 2du d\Omega - 2r\beta_A du dx^A + \gamma_{AB} dx^A dx^B \tag{4.5}$$

on O where $\alpha, \beta, \gamma_{AB}$ are smooth functions on O and on $\mathscr{I}^+ \cap O$ α is a real constant and $\beta_A = 0$. Note that $\gamma_{AB} dx^A dx^B$ is a Riemannian $(d-2)$-metric which does not, in general, coincide with the metric s induced on Σ by $\tilde{g}$ when $\Omega \neq 0 \neq u$.

It is not necessarily obvious that such a coordinate system exists on all of $\mathscr{I}^+$ and not just on $\mathscr{I}^+ \cap O$. If a global coordinate system exists on all of $\mathscr{I}^+$ then $\Sigma(u, \Omega = 0)$ becomes a foliation of $\mathscr{I}^+$ and the coordinates $\mathscr{I}^+ \cap O$ are carried to all of $\mathscr{I}^+$ by imposing that (Ω, x^A) are constant along the orbits of n^a generating $\mathscr{I}^+$ and u can be visualized as the coordinate along $\mathscr{I}^+$. However, in general, one might need more than one coordinate patch to cover $\mathscr{I}^+$. If some null Killing vector exists at null infinity it induces a one-parameter group of isometries and it follows that a global foliation of $\mathscr{I}^+$ exists. While we do not assume that Killing vector fields exist there are asymptotic symmetries (diffeomorphisms preserving the asymptotic structure at infinity), which are tangent to $\mathscr{I}^+$, and play a role similar to a Killing symmetry. This motivates why a global coordinate system in some neighborhood of $\mathscr{I}^+$ might exist. We do not investigate this further and, in particular, do not give a proof for the existence of such a global coordinate system. In the following we instead assume that such a coordinate system exists and add it as a condition for asymptotic flatness at null infinity. Far away from null infinity geodesics may overlap and it is not possible to carry the coordinates further and thus there is no global coordinate system. Summarizing, we have the following definition. An even-dimensional spacetime is called **asymptotically flat** near null infinity if there exists a conformal transformation $M \to \tilde{M}$ in the sense described above, $\mathscr{I}^+$ is isomorphic to $\Sigma \times \mathbb{R}$ where Σ is a compact, $(d-2)$ dimensional manifold, and near null infinity the unphysical metric takes the form (7.4). This definition of asymptotically flat is more general than the usual definition where one demands that the metric becomes asymptotically Minkowski and thus $\Sigma \simeq S^{d-2}$. We opt for a more general definition which will be justified in the following chapter. Finally, denoting by r the physical distance (that is infinity is located at $r = \infty$) and setting $\Omega = 1/r$ the physical metric can be written as

$$r^2\tilde{g} = g = -2\alpha du^2 - 2dudr - 2r\beta_A dudx^A + r^2\gamma_{AB}dx^A dx^B \,. \tag{4.6}$$

We can check consistency by looking at d-dimensional Minkowski space. In spherical coordinates the metric takes the form

$$ds_M^2 = -dt^2 + dr^2 + r^2 s_{AB}dx^A dx^B \tag{4.7}$$

where $s_{AB}dx^A dx^B$ is the metric of the $(d-2)$-dimensional unit sphere. Defining the retarded time as $u = t - r$ (which justifies the name) yields

$$ds_M^2 = -du^2 - 2dudr + r^2 s_{AB}dx^A dx^B \,. \tag{4.8}$$

This is equal to (4.6) in the limit $r \to \infty$, when taking the above conditions into account and if we take $\Sigma \cong S^{d-2}$.

4.1.3 Example: Schwarzschild

As an example we look at the Schwarzschild metric and how it transforms from the standard coordinates to Bondi coordinates. We start in $d = 4$ dimensions, the generalization to higher dimensions is then straightforward. In spherical coordinates the Schwarzschild metric takes the well-known form

$$g_{4d} = -adt^2 + a^{-1}dr^2 + r^2d\sigma^2 \tag{4.9}$$

where $a = (r - c)/r$ is a function for the radius, $c \in \mathbb{R}$ is a parameter and $d\sigma^2$ the line element of the sphere. The retarded time is defined as

$$u = t - r_\star \,, \tag{4.10}$$

where $r_\star$ is the tortoise coordinate defined by

$$r_\star = \int a^{-1}dr = c\ln(r/c - 1) + r\,. \tag{4.11}$$

Thus, we have $r_\star \to -\infty$ as $r \to c$ and

$$\frac{dr_\star}{dr} = a^{-1}\,. \tag{4.12}$$

The reason for defining the retarded time this way and not by, say, $u = t - r$, is that for a geodesic tangent k^a the equation for radial null geodesics is given by

$$g_{ab}k^ak^b = 0 \Rightarrow \frac{dt}{dr} = \pm\frac{r}{r-c} \tag{4.13}$$

and thus $t = \pm r_\star + \text{const}$. So far, these are the usual definitions encountered, for example, in the procedure to analytically continue the coordinates to the whole spacetime (Eddington-Finkelstein coordinates), where one also utilizes the equation for radial null geodesics to define coordinates. For large r the difference between r and $r_\star$ is of order $\ln(r)$ and asymptotically $r \sim r_\star$. Since the Bondi coordinates are only an asymptotic coordinate system, the definitions provided here are coherent with the ones introduced above. We have

$$dt = du + dr_\star \quad \text{and} \quad \frac{dr_\star}{dr} = a^{-1}\,. \tag{4.14}$$

Hence, the metric takes the form

$$\begin{aligned} g_{4d} &= -a(du^2 + a^{-2}dr^2 + 2a^{-1}dudr) + a^{-1}dr^2 + r^2d\sigma^2 = -adu^2 - 2dudr + r^2d\sigma^2 \\ &= -(1 - c/r)du^2 - 2dudr + r^2 s_{AB}dx^A dx^B\,, \end{aligned} \tag{4.15}$$

which is the Schwarzschild metric in Bondi coordinates in four-dimensional spacetime. To find the metric in higher dimensions it is only necessary to modify the parameter a towards $a = 1 - c/r^{d-3}$ and $d\sigma^2$ corresponds now to the metric of an unit $(d-2)$-sphere. For now, this is simply a definition of a new metric. It is not immediately obvious that this is the right generalization to higher dimensions. It was first found by Tangherlini in [45] and we will show below in section 6.1.1, by explicitly solving the Einstein equations, that this is indeed the Schwarzschild solution in d spacetime dimensions. In the new coordinates this metric is

$$g_d = -(1 - cr^{3-d})du^2 - 2dudr + r^2 s_{AB} dx^A dx^B \, . \tag{4.16}$$

Comparing with (4.6) one can see that $\beta_A = 0$ and $\alpha = -(1 - cr^{3-d})$. In the next section, when discussing the definition of mass in general relativity in four dimensions, we will continue with discussing this metric and relate the parameter c to the mass of the Schwarzschild black hole.

4.2 Review of Mass in 4D

While ‘mass’ is a relatively straightforward concept in everyday live and classical mechanics it becomes more difficult in special relativity, where it is famously equal to the energy of a system. Still, the mass appears as a simple parameter in relativistic Lagrangians and there are no conceptual problems. However, this changes when taking gravity into account by going to general relativity. Here, it is a non-trivial problem to even define some concept of mass as there is no way to define a satisfying stress-energy tensor of the gravitational field. In most classical field theories the total energy of a system can be defined as a volume integral over a positive energy density T_{00}. Then the positivity of the total energy is a consequence of the conservation of the stress-energy tensor with a positive timelike component. In general relativity this is impossible. While locally there is a stress-energy tensor encoding the energy density of matter fields, there is none for the gravitational field, since this would violate the equivalence principle. Namely, a freely falling observer does not measure any gravitational field and thus the gravitational energy density cannot be defined at spacetime points. However, matter fields and the gravitational field contribute both to the total energy of a system and hence it is not possible to carry over the definition of total energy of a system as an integral over T_{00}. We have to accept the nonexistence of a local notion of energy density in general relativity. Since a notion of mass/energy would be desirable a different path has to be taken. That is, instead of using local (point-like) definitions it is necessary to look for notions associated with extended domains of spacetime. Then, it is indeed possible to define, e.g., the total energy of an isolated system employing the asymptotic flatness of spacetime in a manner described below. As required, this asymptotic mass/energy definition includes contributions from the matter as well as from the gravitational field. Relating this total energy to the local energy density of matter fields given by the stress-energy tensor is a non-trivial task [46]. However, these definitions exist only if some additional structure, e.g., symmetry, is assumed. We now review how a notion of asymptotic/total energy is defined in four dimensional general relativity. To do this we first look at Newtonian theory and afterwards try to find a fitting analogue in general

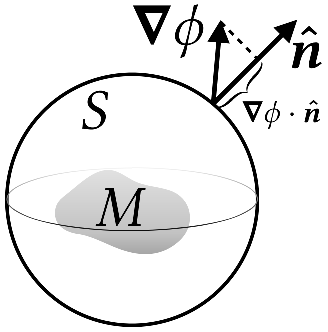


Figure 4.2: To define the mass M in Newtonian physics one considers a sphere S that completely encloses the mass. The projection of the gradient of the gravitational potential, $\boldsymbol{\nabla}\phi$, onto the outward pointing normal $\hat{\boldsymbol{n}}$ of the sphere has to be integrated over S.

relativity. The general idea is that in flat spacetimes, symmetries give rise to conserved integrals and one might hope that in asymptotically flat spacetimes the asymptotic symmetries give rise to similar integrals. This turns out to be indeed the case. For example, asymptotic time translations at null infinity give rise to the so-called Bondi mass. Additionally, one could try to define an energy-momentum for finite regions of spacetime. This leads to many so-called quasi-local mass definitions. We will not discuss this here as it is nicely reviewed for example in [47, 48].

4.2.1 Mass in Newtonian Theory

Consider an isolated system. In the vacuum region outside the system, the Poisson equation for the Newtonian potential ϕ reduces to Laplace's equation

$$\Delta\phi = 0\,. \tag{4.17}$$

Having our goal of defining mass in general relativity in mind, we now choose a slightly unusual way to define the total mass M of the system, namely

$$M_{\text{Newt}} \coloneqq \frac{1}{4\pi}\int_S \boldsymbol{\nabla}\phi\cdot\hat{\boldsymbol{n}}\,dA\,. \tag{4.18}$$

The surface integral is taken over a topological 2-sphere S completely enclosing the source(s) of the gravitational field and $\hat{\boldsymbol{n}}$ is the unit normal of S pointing outward, see Fig. 4.2. Because (4.17) holds the integral is independent of S, since the mass enclosed by the surface does not change. To connect this to the standard definition of the mass, we consider the multipole expansion of

$$\phi(\boldsymbol{r}) = -\int_{\mathbb{R}^3}\frac{1}{|\boldsymbol{r}-\boldsymbol{x}|}d\tilde{m}(\boldsymbol{x})\,, \tag{4.19}$$

with $d\tilde{m}(\boldsymbol{x})$ as the differential mass. The result

$$\phi(\boldsymbol{r}) = -\frac{\tilde{M}_{\text{Newt}}}{|\boldsymbol{r}|} + O(\boldsymbol{r}^{-2}) \tag{4.20}$$

is equal to the work needed to bring a unit mass from infinity to the distance $\boldsymbol{r}$ from a point mass $\tilde{M}_{\text{Newt}} = \int_{\mathbb{R}^3} d\tilde{m}(\boldsymbol{x})$. Thus,

$$\boldsymbol{\nabla}\phi(\boldsymbol{r}) = \frac{\tilde{M}_{\text{Newt}}}{\boldsymbol{r}^2}\hat{\boldsymbol{r}} + O(\boldsymbol{r}^{-3}) \,. \tag{4.21}$$

Plugging this into (4.18), choosing for S the 2-sphere, and working with spherical coordinates yields

$$M_{\text{Newt}} = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1} \boldsymbol{\nabla}\phi \cdot \hat{\boldsymbol{r}}\; r^2 d(\cos\theta)d\varphi \simeq \tilde{M}_{\text{Newt}} \,, \tag{4.22}$$

and thus the two definitions are equivalent. The interpretation of (4.18) is straightforward: $\boldsymbol{\nabla}\phi$ is the force that must be exerted on a unit mass to "neutralize" the gravitational force and hold it at one point. Hence, $4\pi M$ is the total force necessary to hold test matter, with unit surface mass density distributed over S, in place. The definitions in general relativity are similar to the Newtonian case.

4.2.2 Mass in General Relativity

The simplest case is an asymptotically flat spacetime which is static. We always assume that the spacetime is vacuum near infinity and follow the discussion in [35]. There exists a timelike Killing vector ξ^a, which is normalized such that the so-called redshift factor $V := \xi^a\xi_a$ approaches 1 at infinity. In general it is impossible to have a meaningful notion of "staying in place", since there is no background manifold to be used as a reference point. However, for static (and stationary) spacetimes it is possible by virtue of the timelike Killing vector field, namely, an observer is staying in place if they are following an orbit of ξ^a. Thus, the acceleration of the orbit

$$a^b = \frac{1}{V^2}\xi^a\nabla_a\xi^b \tag{4.23}$$

is equal to the force needed to keep a unit test mass in place. This (local) force differs from the force that must be applied by an observer at infinity by a factor of V. Hence, the total outward force that must be exerted by an observer at infinity is

$$F := V \cdot \int_S a_b\hat{n}^b \; dA = \frac{1}{V}\int_S \hat{n}^b\; \xi^a\nabla_a\xi_b \; dA \,, \tag{4.24}$$

where S is again a topological 2-sphere with the unit outward pointing normal $\hat{n}^a$, which is orthogonal to ξ^a, see Fig. 4.3. Let ϵ_{abcd} be the volume element associated with the spacetime

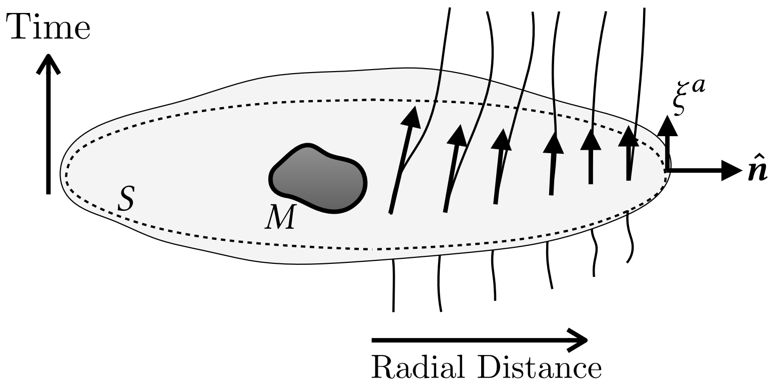


Figure 4.3: If an (asymptotically) Killing vector field ξ^a is present, the mass M present in spacetime can be found by integrating over a sphere S at infinity (dashed line). Following integral curves of a Killing vector field ξ^a is the analogue of "staying in place" in Newtonian physics.

metric. Using Killing's equation we can rewrite (4.24) as

$$F = -\frac{1}{2}\int_S \epsilon_{abcd}\nabla^c \xi^d = -\frac{1}{2}\int_S \star\xi \,. \tag{4.25}$$

It is possible to show that this integral is independent of the choice of S, as was the case for the integral (4.18) in the Newtonian case. Since both integrals, (4.18) and (4.25), are independent of S and they have the same physical interpretation it is natural to define

$$M_{\text{Komar}} := -\frac{1}{8\pi}\int_S \epsilon_{abcd}\nabla^c \xi^d = \frac{F}{4\pi} \,. \tag{4.26}$$

The assumption that ξ^a is a Killing vector is the only requirement to show that the integral is independent of the choice of S and hence (4.26) may also be used as a satisfactory notion of mass for stationary, asymptotically flat spacetimes. This result was first derived in [49] and thus (4.26) is called **Komar mass**. It is a satisfying definition of mass for all stationary, asymptotically flat spacetimes which are vacuum near infinity. Due to the symmetry used in this definition, the Komar mass is the most straightforward generalization of the definition of physical quantities as conserved under symmetry (c.f. Noether theorem charges). A similar construction employing spacelike instead of timelike Killing vectors leads to a definition of angular momentum.

Moving on to non-stationary, asymptotically flat spacetimes the difficulty arises that the notion of "staying in place" used above is no longer available. Thus, it is not obvious how to generalize the above scheme or if it is even possible. It seems that the only chance to define the mass of an isolated system is for an observer to still measure from infinity while replacing the Killing vector with something new. Recall from section 4.1 that there are different kinds of infinities, e.g., spatial infinity and null infinity (move away from the system along space-like or null directions, respectively). It turns out that it is possible to replace the Killing vector and there are, corresponding to the two kinds of infinity, two types of masses one can define. These are the Bondi mass at null infinity and the ADM mass (Arnowitt, Deser, and Misner) at spatial infinity. We will see below why such a distinction/difference was not necessary for the Komar mass. First, we look at the Bondi mass (sometimes also called Trautman-Bondi mass or Trautman-Bondi-Sachs mass). It was first introduced by [50] and [11, 42, 43], where to each null cone a number, the so-called Bondi mass of this null cone, was associated. That is, to define this mass one looks at a fixed retarded time $u = t - r$ by going to null infinity on a asymptotic null surface. For each fixed u there is then a number which quantifies the mass/energy in a system at time u. This is the total mass on the chosen hypersurface. We now want to sketch how this mass is defined explicitly. Note that there are two (equivalent) “languages” in which an expression for the Bondi mass can be formulated. Owing to the similarity to the Komar mass we start with the definition of Bondi mass using the so-called linkage formulation which was introduced by [51–53] building on work of [38]. Let ξ^a now be the generator of an asymptotic time translation symmetry[1]. In particular, ξ^a does not satisfy the Killing equation everywhere on the spacetime but only at infinity. Hence, in general, an integral similar to (4.26) now depends on the choice of S (this being the reason why the Komar mass cannot be used as a mass definition). But, since ξ^a fulfils the Killing equation at infinity, ξ^a acts more and more like a Killing vector the closer it is to infinity. As a consequence, the dependence of the integral on S becomes smaller and smaller as one goes to infinity. Exploiting this, we can slightly modify (4.26) to adapt it to the new situation. Thus, define the mass as

$$M(u) := -\frac{1}{8\pi}\int_S \epsilon_{abcd}\nabla^c\xi^d = -\frac{1}{8\pi}\int_S \star\xi \tag{4.27}$$

where S is an asymptotic two-sphere at given retarded time u. This expression is not inherently gauge invariant and a gauge condition needs to be added. As shown by [58] a satisfactory choice is $\nabla_a\xi^a = 0$ which is in particular fulfilled if ξ^a satisfies the Killing equation at infinity. With this choice, while the integrand is not invariant under passage to an equivalent generator, the integral is invariant. Note that there is also the symplectic, or Hamiltonian, approach where the

[1]An asymptotic symmetry is a spacetime diffeomorphism that preserves the asymptotic structure of the spacetime. Asymptotic symmetries form a group (Spi group at spatial infinity and BMS group at null infinity) which includes the Poincaré group as a subgroup. See, e.g., [35, 42, 54–56] (BMS group) and [39, 40, 57] (Spi group) and references therein for an introduction and discussions of this topic in $d = 4$ and, for example [15, 17] for higher dimensions.

mass is defined as the quantity conserved under infinitesimal asymptotic timelike translations[2], see, e.g., [56, 59–62] for an introduction. The relation between the linkage and the symplectic approaches was examined by [63] essentially showing that, in physically relevant cases, the approaches yield the same results.

The second "language" is the way it was originally defined by Bondi, Sachs and others in [11, 42, 43] using Bondi coordinates, that is, the metric takes the form (4.6). As a first step/as motivation we will try to use similarities with Newtonian physics again. We look at a system with mass M such that its gravitational potential is $\phi = -M/r$, which defines the acceleration by taking the gradient, $\mathbf{a} := -\boldsymbol{\nabla}\phi$. The Newtonian limit of general relativity is given by the following conditions [35, 64, 65]:

- Particles move slowly in the sense that $\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}$ where τ is the proper time of a particle.
- The gravitational field is a perturbation h of Minkowski space η, i.e., $g = \eta + h$.
- The gravitational field is static, $\partial_t g = 0$.

Using these three assumptions the geodesic equation of slow particles simplifies considerably, namely

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = \frac{d^2x^\mu}{d\tau^2} - \frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00}\left(\frac{dt}{d\tau}\right)^2 = 0\,. \tag{4.28}$$

Due to the assumption that the gravitational field is static, the 0-component of this equation yields $dt/d\tau = const$. The i-component is equal to

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 \partial_i h_{00} \tag{4.29}$$

which is reminiscent of Newtons second law and thus we identify $h_{00} = -2\phi$ or $g_{00} = -(1+2\phi) = 2M/r - 1$. Hence, we see that for consistency there should in general be a relation between the time-component of the metric and the mass of the system. However, in general relativity the mass of a system can change by radiating gravitational waves and, in general, an expression for the mass of a system must depend on time. Since there are no gravitational waves in Newtonian gravity it is clear that we cannot find a satisfactory notion of mass by considering only the Newtonian limit. Thus, in Bondi coordinates, we can use $g_{uu} = -2\alpha$ to define the mass of a system at a fixed $u = u_0$ but a time-dependent component is still missing. That is, let u_0 be fixed and let (θ, φ) be the usual spherical coordinates on the unit sphere at infinity. Then integrating

$$m(u_0) := \lim_{r\to\infty} r^2 \frac{\partial\alpha}{\partial r}\bigg|_{u=u_0} \tag{4.30}$$

[2]This approach will be relevant later in the context of higher dimensional Bondi mass.

over a sphere S at infinity, $\int_S m(u_0)\sin(\theta)d\theta d\varphi$, defines the mass of a system at time u_0. Taking u_0 as the starting point of the observation this mass can be viewed as the "starting mass" previous to any radiation being emitted. Therefore, we are now looking for a function of $m(u)$ telling us how the mass evolves over time with $m(u_0)$ as an initial value. As shown by [42, 43] a so-called **news function** can be defined which completely determines the change of mass. Following them we define the news tensor as

$$N_{AB} := \frac{1}{2}\frac{\partial}{\partial u}\left[\lim_{r\to\infty} r\left(\gamma_{AB} - s_{AB}\right)\right]. \tag{4.31}$$

This quantity describes how the deviation of the spherical part of the general metric from the unit sphere changes over time near infinity, that is, it encodes the energy flux of gravitational waves, since gravitational radiation appears as a perturbation of the AB-part of the metric. Knowledge of this tensor is sufficient to know how the mass changes over time since, using the Einstein equations, one can show [44] that

$$2\partial_u m = \mathcal{D}_A\mathcal{D}_B N^{AB} - N_{AB}N^{AB}, \tag{4.32}$$

where $\mathcal{D}_A$ is the covariant derivative of s_{AB}. Assuming N_{AB} is known for $u_0 \leq u \leq u_1$ we can integrate this equation with initial value $m(u_0)$ yielding a time-dependent function $m(u)$. Finally, integrating $m(u)$ over a sphere at infinity, defines the Bondi mass. Thus, we have the following result. Let S be the 2-dimensional unit sphere at infinity which is parametrized by (θ, φ). The time-dependent scalar function

$$M(u) := \frac{1}{4\pi}\int_S m(u,\theta,\varphi)\sin\theta d\theta d\varphi \tag{4.33}$$

is called the **Bondi mass**. Integrating (4.32) one finds the **Bondi mass loss formula**

$$\frac{d}{du}M(u) = -\frac{1}{4}\int_S N^{AB}N_{AB}\sin\theta d\theta d\varphi. \tag{4.34}$$

Since the right-hand side of (4.34) is non-positive, the Bondi mass can only decrease or stay constant, the latter is the case only if there is no news. It was shown by [52, 53] that the two ways of writing the Bondi mass in the different "languages" described, i.e., via the linkage formalism or metric coefficients, are indeed equivalent. In fact, the definition is unique in some sense, see [66].

Example: Schwarzschild

As a simple example we look at the Schwarzschild metric. As we have seen, we can define the mass either through an integral or through a power series of some metric component. Since the

Schwarzschild metric is static, the Komar mass and the Bondi mass (as well as the ADM mass discussed next) are equal. Thus, we start with computing (4.26)/(4.27) for the Schwarzschild metric, which reads, in spherical coordinates,

$$ds^2 = -\left(1 - \frac{c}{r}\right) dt^2 + \left(1 - \frac{c}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.35}$$

The timelike Killing vector is given by

$$\xi_\mu = \left(\frac{c}{r} - 1, 0, 0, 0\right) \tag{4.36}$$

and thus we have

$$\alpha \equiv \epsilon^{ab}\epsilon_{abcd}\nabla^c\xi^d = -\frac{1}{\sin\theta} r^2 \sin(\theta)\frac{c}{r^2} = -c. \tag{4.37}$$

Plugging this into (4.26) yields

$$M_{\text{Komar}} = -\frac{1}{8\pi}\int_S \alpha\epsilon_{ab} = -\frac{1}{8\pi}\int_0^{2\pi}\int_0^{\pi} c\sin(\theta)d\theta d\phi = \frac{c}{2} \tag{4.38}$$

for the mass of a Schwarzschild black hole. To test the second way the Bondi mass was defined we need the Schwarzschild metric in Bondi coordinates, i.e., (4.15), where we immediately see that the uu-component is equal to $-2\alpha = c/r - 1$ and thus, using (4.30), we find $M = c/2$ again. Since the metric is static, the news tensor N_{AB} is equal to zero. Hence, the Schwarzschild metric can be rewritten in the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{4.39}$$

which is usually found by taking the Newtonian limit to relate c and M since, as we have seen, this is equal to the definition of the Bondi mass for the static Schwarzschild metric. The case of the Kerr metric is very similar.

The second case, the ADM mass, is defined at a fixed physical time t at spatial infinity. It was introduced and investigated by R. Arnowitt, S. Deser and C. W. Misner in a series of papers [10, 67, 68], see [69] for a contemporary review of these and related papers by the same authors. The original approach was in the Hamiltonian framework ([70, 71]) a reformulation similar to the linkage formalism described above exists as well, see, e.g., [39–41, 48, 55, 72] and references therein. We will only discuss this topic heuristically and not go into details. The original result by ADM can be summarized as follows. Consider 3-slices F_t where the family $\{F_t\}$ is a foliation of the spacetime such that on every slice the time t is constant. To each slice F_t we can associate a number, the ADM mass. Let t^μ be the evolution vector, a time-like vector which is normal to F_t, which is chosen such that it generates asymptotic time translations. The ADM energy is

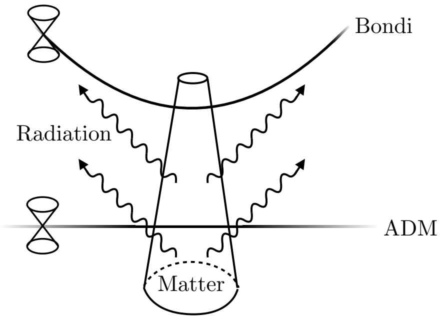


Figure 4.4: Sketch of matter in a compact region of spacetime which emits gravitational radiation. The ADM mass is defined on a spacelike surface F and thus all radiation crosses this surface eventually. However, the Bondi mass is defined on an asymptotically null surface N_u at time u and thus there might be radiation, which was emitted before time u, that never crosses N_u and therefore does not contribute to the Bondi mass on this slice.

defined as the surface integral over the asymptotic behavior of the gravitational field and it can be shown that an expression in asymptotically cartesian coordinates is given by [35, 47]

$$E = - \lim_{r\to\infty}\Big|_{t=\text{const.}} \frac{1}{16\pi} \int_S \left(\partial^k h_{ik} - \partial_i h\right) dA^i \tag{4.40}$$

where dS^i is the normal surface element to a sphere of constant r, $h_{ik} = g_{ik} - \eta_{ik}$ and $h = \eta^{ik} h_{ik}$. The integral is taken over a bounding surface in the asymptotically flat region of F_t. It can be shown that for any foliation $\{F_t\}$ where t^μ coincides with k^μ at infinity the Komar mass is equal to the ADM mass.

The above definitions give rise to a natural interpretation of the two masses, see also the sketch in Fig. 4.4. Since the ADM represents the net energy crossing an asymptotically flat spacelike surface F, this surface eventually intersects all emitted radiation, because no signal can travel faster than light, and therefore it is not possible for any physical process to change the asymptotic behavior at spatial infinity. Hence, the ADM mass is time independent and represents the total energy of the system. On the other hand, the Bondi mass represents the energy crossing an asymptotically null surface N_u and N_u does not intersect all emitted radiation, which results in a time-dependence of the Bondi mass and the quantity is interpreted as the mass remaining in the system at time u. Thus, while the ADM mass is a scalar constant over time the Bondi mass is a scalar function of u which can change due to gravitational radiation. As shown in [73], this intuition is right and for physically interesting systems the difference between the Bondi mass and the ADM mass is the energy flux carried away by the radiation between infinite past and given retarded time. Another question that arises is whether the masses defined above have to be positive for nonsingular spacetimes. In the following section we will explain why this is an issue and sketch the proof(s) of the so-called positive energy theorem.

4.3 Positivity of Mass in General Relativity in 4D

Again, we start by looking at Newtonian gravity. Here, systems with negative energy are ubiquitous since any bound system has negative total energy. Since the gravitational potential is unbounded from below, it is possible to construct systems with negative total energy even if the rest mass of the matter involved is taken into account. However, problems occur if we go to general relativity. First, note that Minkowski space fixes the ground state with zero energy so we are not free to choose any arbitrary zero point for the energy. Assume now that a system with negative total energy constructed in Newtonian physics also exists in general relativity. In this case, since energy is equal to mass, this would mean that the system has negative gravitational mass which would result in a repelling rather than attracting gravitational force. Additionally, one can imagine a radiating system. Since the radiation will carry away positive energy the total energy in the system will decrease. Since the total energy was negative in the beginning and if the energy were in fact unbounded from below it would be possible to extract an unlimited amount of energy which is clearly unphysical. Thus, if systems with negative total mass were allowed in GR this would indicate a fundamental problem with the theory in the sense that there might not be any stable solutions. The idea from a physics point of view why this situation (that is, total negative energy) cannot occur is that if we were to create a bound system with large negative total energy we would inevitably end up with a black hole forming[3] which has positive total energy and this "saves" the theory. Thus, it is possible only in Newtonian gravity to create a system with arbitrarily negative energy, but there we do not have mass-energy equality or gravitational radiation and hence no problem.

While the intuitive solution is relatively straightforward, it has proven remarkably difficult to actually establish a proof of mass positivity in general relativity. The reason that there are issues for general relativity, but not for most other theories is due to the nonexistence of a stress-energy tensor for the gravitational field, as discussed in the beginning of section 4.2. It is usually a trivial consequence of the conserved stress-energy tensor $T_{\mu\nu}$ with positive timelike component, which ensures that a physical system cannot radiate away more energy than it initially had and that the ground state is stable. Since, as discussed above, the definition of the total mass of a system is (more or less) independent of the stress-energy tensor, the standard

[3] There are in fact some connections between the proofs of the famous singularity theorems and proofs of the positive energy theorem, see e.g. [74].

arguments to show positivity of mass are no longer applicable. Thus, the stability of Minkowski space as a ground state and a proof of the positive energy theorem are far from obvious and highly non-trivial. This is the reason why the first proof was published only in 1979 by R. Schoen and S.-T. Yau who showed in [13] that the ADM mass is positive, a more general case with less strict assumptions was shown by the some authors shortly afterwards [75, 76]. Already earlier, in 1977, S. Deser and C. Teitelboim showed that the energy in supergravity is positive [77] using the Hamiltonian formalism of supergravity [78]. Following a suggestion of M. T. Grisaru ([79]) E. Witten considered the $\hbar \to 0$ limit of the proof in supergravity (in this limit the fermionic parts present in supergravity drop out and classical general relativity is recovered) and found a significantly simpler proof of the positivity of ADM mass [14]. The new idea (the crucial inspiration drawn from supergravity) was to use spinors also in classical general relativity to facilitate the proof. To understand how spinors come into play it is advantageous to briefly look at the definition of charge in classical electromagnetism. We have the current 1-form J and the electromagnetic field tensor 2-form F. They are related by the Maxwell equation $d \star F = \star J$. Let V be a volume containing some charge distribution. The total charge in V is given by

$$q = \frac{1}{8\pi} \int_V \star J \tag{4.41a}$$

which, using the Maxwell equation $d \star F = \star J$, can also be written as

$$q = \frac{1}{8\pi} \int_V d \star F = \frac{1}{8\pi} \int_{\partial V} \star F \tag{4.41b}$$

where Stokes theorem was used in the last step. Now, note the similarity between (4.41a) and the mass definition (4.27), but keep in mind that we are considering the ADM mass at the moment. In the definition of the mass, the 1-form is obtained from the asymptotic timelike Killing vector by lowering the index. The question is whether there is also an analogue of (4.41b), i.e., whether there is a 2-form which, integrated over infinity, yields the mass. Let us take the analogue with electromagnetism and the similarity of (4.41a) and (4.27) as motivation to look for an appropriate 2-from E. It seems that there is not much freedom in defining E. A dimensional analysis shows that there must be one derivative involved and we integrate over infinity where the only distinguished vectors are the asymptotic symmetries. Thus, one might guess that there are only two possibilities, namely [80]

$$E^1_{ab} = \nabla_{[a}\xi_{b]} \, , \tag{4.42a}$$

and

$$E^2_{ab} = \epsilon_{abcd} \nabla^c \xi^d \, . \tag{4.42b}$$

However, both choices are unsatisfactory. If we look at the integral $\int_{\partial S} \star E$ over a two-sphere the "mass" resulting from E^1_{ab} depends on the order $1/r$ part of ξ^a and thus the result is not related to the mass in the spacetime, cf. the Newtonian limit (4.29) and the discussion of the Schwarzschild metric in section 4.2.2. The integral over E^2_{ab} vanishes due to Stokes theorem [80]. Thus, the 2-forms which can be formed from ξ^a do not yield a satisfactory definition of mass and it seems that there is no analogue of (4.41b). However, now the motivation from supergravity comes into play. If one allows spinors to be used, there are two additional possibilities. They are constructed such that their imaginary part is equal to (4.42a) and (4.42b), respectively, but the real part differs and thus is the interesting part. Take a spinor Ψ and define

$$E^3_{ab} = \nabla_{[a}(\bar{\Psi}\gamma_{b]}\Psi) \tag{4.42c}$$

$$E^4_{ab} = 2\epsilon_{abcd}\left(\bar{\Psi}\gamma^c\nabla^d\Psi - \nabla^d\bar{\Psi}\gamma_c\Psi\right) . \tag{4.42d}$$

The integral over $\Re(E^3_{ab})$ has the same issue as the one over E^1_{ab}. However, E^4_{ab} leads to an expression depending only on the asymptotic value of Ψ and in fact let $\Psi = \Psi_0 + O(r^{-1})$, where Ψ_0 is Killing. The vector $\bar{\Psi}_0\gamma^a\Psi_0$ is equal to the Killing vector ξ^a (c.f. (3.35)). Setting $E = E^4$ we have

$$\frac{1}{8\pi}\int_S \star E \tag{4.43}$$

which is real (since the contribution of E^2 vanishes under the integral) and this expression is the analogue of (4.41b). This 2-form is called **Witten-Nester 2-form**, since it was not introduced in Witten's original proof in [14] but only introduced by J. A. Nester in [81], where a small error in Witten's line of argument was corrected. It was shown by [14, 81] that (4.43) is asymptotically equal to the definition of the ADM and thus the expression (4.43) is an equivalent expression for the ADM mass. Therefore, to show positivity of mass it suffices to show that (4.43) is positive. Using Stokes theorem, one thus wants to show that $\nabla^a E_{ab} \geq 0$. Assuming only that the dominant energy condition (a local condition) is fulfilled and restricting the freedom of Ψ by allowing only spinors, which fulfil the Witten equation $\gamma^i\nabla_i\Psi = 0$, where the index i runs only over spatial coordinates, positivity can been shown, using some standard identities for gamma matrices and spin derivatives, rather easily, see [14, 81]. (We will come back to this below in our proof of the positivity in higher odd dimensions.) Having shown that the ADM mass is positive, the natural next step was to consider the Bondi mass. Schoen and Yau were able to adopt their strategy to the Bondi mass [82] while W. Israel and J. M. Nester [83], M. Ludvigsen and J. A. G. Vickers [84, 85], as well as G. T. Horowitz and M. J. Perry [86] used Witten's style of argument, with only small modifications, to proof the positivity of Bondi mass in 1981, see also [80]. Witten's argument in particular has afterwards been used to include more general cases, for example spacetimes with black holes [87, 88], electromagnetic charges [89, 90] (see also [91]), and spacetimes which are asymptotically AdS [87]. There are also further investigation of the relation between the proof in classical general relativity and supersymmetry (e.g. [92]) and attempts to make the results more mathematically rigorous by, e.g., [93–95].

The goal of this thesis is to show that the positivity of Bondi mass also holds for arbitrary odd dimensions $d \geq 5$. We do this with with an argument similar to the one in Witten's original work but new complications occur in higher dimensions requiring additional steps. In particular, the spinor we use is not of the simple leading-order type as in four dimensions. Furthermore, while the ADM mass is readily generalized to higher dimensions [70] it turns out that defining the Bondi mass in higher dimensions is considerably more difficult and, in particular, the linkage formalism does not carry over to higher dimensions. We will discuss this in the next two chapters and come back to the positivity afterwards.

Part II

Bondi Mass and Positivity

CHAPTER 5

Assumptions, Setup and Notations

The second part of this thesis is concerned with odd dimensional spacetimes and the goal is to find a coordinate expression for the Bondi mass and show that it is positive. In this chapter we state the assumptions and conditions we impose on the spacetime we are working with as well as fix some notations for the rest of the thesis. It was already mentioned at several points that there are some differences between even and odd dimensions concerning the definition of null infinity. In the first section of this chapter we look at gravitational waves in odd dimensions to illustrate the issue and motivate our assumptions in the second section. These assumptions are in particular concerned with defining asymptotic flatness (since in our case no smooth conformal null infinity is present) by imposing Bondi coordinates. Finally, the most important results of the subsequent chapters are summarized.

5.1 A First Glance at Odd Dimensions: Gravitational Waves

At first glance it is not at all obvious why there should be a difference between even and odd dimensional spacetimes in general. While one might expect a difference between $d = 3$ and $d = 4$ dimensional spacetimes it is not clear why there should be a difference between, say $d = 8$ and $d = 9$. However, it has been shown (already around 1900 by Hadamard for the scalar wave equation in flat space [96, 97]) that there are crucial differences between the two cases. One of the simplest and most instructive examples where the difference becomes apparent are gravitational wave. (In fact, it would already suffice to look at the scalar wave equation in Minkowski spacetime to see the difference.) To see this it suffices to look at the Green's functions. In essence, Huygen's principle is violated already at leading order in r in odd dimensions and the waves have a "tail", i.e., support inside the lightcone. Before comparing both cases we quickly recapitulate the standard setup for the treatment of gravitational waves in linearized gravity. An introduction to this topic can be found in any standard book on general relativity, see e.g. [35, 64, 65].

We want to derive a metric in the presence of gravitational waves in linearized gravity. Assume that there is some matter with stress-tensor T on a Minkowski background $(\mathbb{R}^d, \eta)$. The trace-reversed metric perturbation is defined as

$$\bar{h}_{ab} := h_{ab} - \frac{1}{2}\eta_{ab}h\,, \tag{5.1}$$

where $h \equiv \eta^{ab}h_{ab}$. Working in the gauge $\partial^a \bar{h}_{ab} = 0$ the linearized Einstein equation reads

$$\eta^{cd}\partial_c\partial_d\bar{h}_{ab} = 16\pi T_{ab}. \tag{5.2}$$

The solution of this equation is given by

$$\bar{h}_{ab} = 16\pi \int d^5x' G(x,x')_{ab}^{a'b'} T_{a'b'}(x') \tag{5.3}$$

where $G(x,x')_{ab}^{a'b'} = \eta_a^{a'}\eta_b^{b'}G(x,x')$ and $G(x,x')$ is the scalar Green's function of the inhomogeneous wave equation for a field on d-dimensional Minkowski spacetime $(\mathbb{R}^d, \eta)$, i.e.,

$$\partial^a\partial_a G(x,x') = \delta(x,x') \tag{5.4}$$

The Green's function is found to be [98–100]

$$G_e(x,x') = \frac{1}{2}(2\pi)^{\frac{2-d}{2}}\theta(t-t')\left(-\frac{1}{|\xi|}\frac{\partial}{\partial|\xi|}\right)^{d/2-2}\left(\frac{\delta(t-t'-|\xi|)}{|\xi|}\right) \tag{5.5}$$

in even dimensions while in odd dimensions it is

$$G_o(x,x') = (2\pi)^{\frac{1-d}{2}}\left(-\frac{1}{t-t'}\frac{\partial}{\partial t'}\right)^{\frac{d-3}{2}}\frac{\theta(t-t'-|\xi|)}{\sqrt{(t-t')^2-|\xi|^2}} \tag{5.6}$$

where $\xi = \boldsymbol{x} - \boldsymbol{x}'$.

We now look more closely at the linearized perturbation h_{ab} in an odd-dimensional Minkowski background. To illustrate what happens at lowest order in r, we look at 5-dimensional spacetime. The treatment for higher[1] odd dimensions is similar. We consider, similar to calculations in [16, 101], a perturbation generated by particle scattering of massive particles, where the ingoing and outgoing particles interact only at a single point that is taken to be the origin. The stress-energy tensor of the outgoing particles in coordinates $(t, \boldsymbol{x})$ is

$$T_{ab}^{(\text{out})}(\boldsymbol{x}) = \sum_i m^i v_a^i v_b^i \delta(\boldsymbol{x} - \boldsymbol{y}^i(t))\frac{d\tau^i}{dt}\theta(t) \tag{5.7}$$

where m^i is the rest mass, $\boldsymbol{y}^i$ the spatial trajectory and v_a^i the four velocity of the ith outgoing particle, τ^i is its proper time. The expression for the ingoing particles $T_{ab}^{(\text{in})}$ has the same form with $t \to -t$ and the total stress-energy tensor is

$$T_{ab} = T_{ab}^{(\text{in})} + T_{ab}^{(\text{out})}. \tag{5.8}$$

[1] There is no gravitational radiation in $d = 3$ dimensions.

In the following, we consider the case of a single particle ($i = 1$) of unit mass ($m^1 = 1$) which is "created" at the origin and rests there, i.e. $\boldsymbol{y}^1 = 0$. This is done to keep the expressions shorter and focus on the aspects we are interested in; the general case can easily be recovered by repeating the following calculation with the full stress-energy tensor (5.8), see also [16, 101]. Plugging (5.6) into (5.3) yields

$$\bar{h}_{ab} = 16\pi \int d^5x' T_{ab}(x') \frac{1}{(2\pi)^2} \left(-\frac{1}{t-t'} \frac{\partial}{\partial t'} \right) \frac{\theta(t-t'-|\xi|)}{\sqrt{(t-t')^2 - |\xi|^2}}. \tag{5.9}$$

Now, substituting (5.7) (for the single particle described above and $v^1_a \equiv v_a$, $m^1 \equiv m$), we have

$$\begin{aligned} &= -\frac{4mv_a v_b}{\pi} \int dt' d^4x' \frac{\delta(\boldsymbol{x}')\theta(t')}{t-t'} \frac{\partial}{\partial t'} \frac{\theta(t-t'-|\xi|)}{\sqrt{(t-t')^2 - |\xi|^2}} \\ &= -\frac{4mv_a v_b}{\pi} \int_0^\infty dt' \frac{1}{t-t'} \frac{\partial}{\partial t'} \frac{\theta(t-t'-|\boldsymbol{x}|)}{\sqrt{(t-t')^2 - |\boldsymbol{x}|^2}} . \end{aligned}$$

Defining $r = |\boldsymbol{x}|$ and $u = t - r$ and using the latter to replace t,

$$\begin{aligned} \bar{h}_{ab} &= -\frac{4mv_a v_b}{\pi} \int_0^\infty dt' \frac{1}{u+r-t'} \frac{\partial}{\partial t'} \frac{\theta(u-t')}{\sqrt{(u+r-t')^2 - r^2}} \\ &= \frac{4mv_a v_b}{\pi} \int_0^\infty dt' \frac{\partial}{\partial t'} \left(\frac{1}{u+r-t'} \right) \frac{\theta(u-t')}{\sqrt{(u+r-t')^2 - r^2}} \\ &- \frac{4mv_a v_b}{\pi} \frac{1}{u+r-t'} \frac{\theta(u-t')}{\sqrt{(u+r-t')^2 - r^2}} \Bigg|_{t'=0}^{\infty} . \end{aligned} \tag{5.10}$$

We are interested in the limit $r \to \infty$ as $u = \text{const.}$ (null infinity). The first term in (5.10) decays as $r^{-5/2}$ while the non-vanishing boundary term in the second term decays as $r^{-3/2}$ and therefore this is the relevant leading-order term. The result is

$$\bar{h}_{ab} = \frac{4mv_a v_b}{\sqrt{2}\pi} \frac{\theta(u)}{\sqrt{u}} r^{-3/2} + O\left(r^{-5/2}\right). \tag{5.11}$$

Doing the trace reverse yields

$$h_{ab} = \frac{4m}{\sqrt{2}\pi} \left(v_a v_b + \frac{1}{d-2} \eta_{ab} \right) \frac{\theta(u)}{\sqrt{u}} r^{-3/2} + O\left(r^{-5/2}\right) . \tag{5.12}$$

By the same procedure the result can be found for all odd dimensions $d \geq 5$ and the leading term in the metric perturbation is of order

$$h_{ab} \sim r^{-d/2+1} + O\left(r^{-d/2}\right) . \tag{5.13}$$

Therefore, the relevant term is of half-integer order in $1/r$. This is in contrast to even dimensions where the perturbation is of integer order, as can be seen when repeating the above computations with (5.5) instead of (5.6) [101, 102]. Note that Huygens' principle (the wave function has support only on the light cone and all modes travel with speed of light) holds for all even dimensions $d \geq 4$ but does not hold for odd dimensions $d \geq 5$, see also [103]. Instead, the wave has a so-called tail meaning that there is support inside the lightcone and some modes travel slower than speed of light, already in the leading order term. This means that a local event has effects at all times. This is due to the function $\theta(u)$ in (5.12), which appears in odd dimensions while in even dimensions it is replaced by a delta distribution. Thus, one sees already at this stage that there are crucial differences between even and odd dimensions. The differences will be important in the next section when an asymptotic expansion of the metric coefficients in Bondi coordinates is introduced.

5.2 Assumptions and Results

We state all assumptions and the general setup in this section and give an overview over the main results we will derive in the subsequent chapters. $(\mathscr{M}, g)$ is a smooth Lorentzian spin manifold of odd dimension $d \geq 5$. g has signature $(-, +, \ldots, +)$. We saw in chapter 4 how to construct Bondi coordinates in even dimensions and how they are used to define asymptotic flatness. Now, we are interested in spacetimes with odd dimensions. As mentioned above it is not possible to define a smooth conformal null infinity since the unphysical metric is at most $(d-3)/2$ times differentiable. But it is still possible to define asymptotic flatness by requiring that suitable Bondi coordinates exist [104, 105]. The idea behind the coordinates remains the same, that is, we want to define a coordinate system far away from the source of gravitational force. Here, "far away" can be visualized as saying that the gravitational force is so weak that massless dust travelling on null geodesics is not influenced significantly by the gravitational force on the time scale at which we are looking at the problem. Then, in this region the geodesics do not intersect, as required in chapter 4. The physical interpretation of the coordinates (u, r, x^A) remains the same as above and so does their "character" (null/timelike/spacelike) and "transportation properties". For example, the surfaces $u = \text{const.}$ are null, $(\partial_a u)(\partial_b u) g^{ab} = 0$, and the $d-2$ scalar functions x^A and r are defined such that they are constant along the integral lines of $(\partial/\partial u)$. Similarly, (u, x^A) are constant along a given geodesic with affine parameter r, see chapter 4. Choosing a fixed retarded time u and going to $r \to \infty$ along the geodesic determined by u we arrive at a "point" and the set of all such points (choosing different u) is called future null infinity. We require the existence of such a coordinate system as a condition for asymptotic flatness such that in the range where the unphysical metric is differentiable the definition agrees with the one given for even dimensions. Additionally, we need the following condition. Assume that there are functions α, β_A, and γ_{AB} of (u, r, x^A) which are smooth and can be expanded in the power series

$$\alpha(u, r, x^A) \sim \sum_{n\geq 0} \frac{\alpha^{(n)}(u, x^A)}{r^{n/2}}\,, \tag{5.14a}$$

$$\beta_A(u, r, x^A) \sim \sum_{n\geq 0} \frac{\beta^{(n)}(u, x^A)}{r^{n/2}}\,, \tag{5.14b}$$

$$\gamma_{AB}(u, r, x^A) \sim s_{AB} + \sum_{n\geq 1} \frac{\gamma^{(n)}_{AB}(u, x^A)}{r^{n/2}}\,, \tag{5.14c}$$

where $n \in \mathbb{N}$ and $\alpha^{(0)}$ is a real, positive constant. As we saw in the last section, there is a dimension-dependent difference in the power of the leading order term for gravitational waves, namely in even dimensions it is an integer power of r while in odd dimensions it is a half-integer power. We do not want to exclude gravitational waves and thus an asymptotic expansion has to be chosen which is consistent with the metric in the presence of gravitational radiation, namely an expansion in half-integer powers of r. This motivates our choice of the power of r in (5.14). A similar ansatz, but with more assumptions, was proposed by [105] and also used in [104, 106, 107]. The power series (5.14) are assumed to exist and to be well defined such that they are, e.g., differentiable, and $\alpha^{(n)}$, $\beta^{(n)}$, $\gamma^{(n)}_{AB}$ are smooth functions of u and x^A but are independent of r.

The spacetime $(\mathscr{M}, g)$ is assumed to be vacuum near infinity, that is, the vacuum Einstein equations hold there. Furthermore, we take it to be asymptotically flat[2], i.e., we require that Bondi coordinates (u, r, x^A) with the properties described above exist and that near infinity the metric has the form

$$g_{ab}dx^a dx^b = -2\alpha du^2 - 2du dr - 2r\beta_A du dx^A + r^2\gamma_{AB}dx^A dx^B\,, \tag{5.15}$$

where the coefficients α, β_A and γ_{AB} are smooth and can be expanded as above. We denote the r-derivative of some quantity x by $\dot{x} \equiv \partial_r x$ and the u-derivative by $x' \equiv \partial_u x$. The compact $(d-2)$-dimensional manifold $\Sigma(u, r)$ is defined near infinity as the surface of constant u and r and we require that a spin structure can be defined on it. Thus, the coordinates on Σ are x^A. In d-dimensional Minkowski spacetime Σ is spherical, $\Sigma \cong S^{d-2}$, and x^A are the usual angular coordinates. The metric on Σ induced by g is the Riemannian metric s_{AB}, which is equal to γ_{AB} at $u = 0$ at infinity. The spin derivative and Levi-Civita derivative on (Σ, s) are denoted $\mathcal{D}$, the ones on $(\mathscr{M}, g)$ by ∇ and the Levi-Civita derivative of γ_{AB} by D. We require that (Σ, s) admits a real Killing spinor ϵ with constant λ. Thus, Σ is an Einstein space. Further, we require

[2]Recall that our definition of asymptotic flatness is used in a slightly different sense than usual. It refers not only to the case that the metric approaches the Minkowski metric at infinity but is slightly more general.

that the metric is related to the Killing spinor by[3] $\alpha^{(0)} = \lambda^2/2$. We can define the conformal transformation $\tilde{g} = \Omega^2 g$ with $\Omega = 1/r$ such that the boundary of $\tilde{M}$ contains future null infinity $\mathscr{I}^+ \cong \mathbb{R} \times \Sigma$, but it is not smooth [18]. Spacetime indices are raised/lowered by s_{AB} unless otherwise noted.

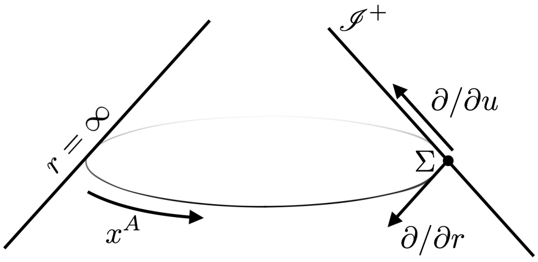


Figure 5.1: A sketch of the Bondi coordinates, a hypersurface and Σ in a conformal diagram. Σ is represented by a point near/at null infinity.

The main results of each chapter are as follows. In chapter 6 we use the Einstein equations to show that coefficients corresponding to low orders of r vanish. The results are summarized in 6.1. We then use these results to derive an explicit expression for the Bondi mass in odd dimensions in Bondi coordinates from results of [17]. In chapter 7, following [19], we establish the main result of this thesis. With some additional assumptions (not yet stated here) we show that the Bondi mass is non-negative, see Theorem 7.2.

[3]We can have Minkowski spacetime at infinity only if $\alpha \sim 1/2$ and if $\lambda \neq 1$ we have a non-flat metric at infinity. We refer to the spacetime as asymptotically flat in either case which is the more general usage of the term mentioned above.

Einstein Equations and Bondi Mass

In this chapter, we want to derive/define an explicit expression for the Bondi mass in odd dimensions $d \geq 5$ in Bondi coordinates and interpret the result in physical terms, i.e., we want generalizations of (4.33) and (4.34). Therefore, we first need to investigate Bondi coordinates in odd dimensions. In the first section, we plug the asymptotic expansions of the Bondi coordinates into the Einstein equations. It turns out that this method simplifies the equations significantly and they can be solved iteratively at each order of r, which reveals some structure in the asymptotic expansion. Then, in the second section, a geometric expression for the Bondi mass derived by [17] in the Hamiltonian framework is utilized to find an explicit expression for the Bondi mass.

6.1 Asymptotic Expansion and Einstein Equations

We assumed that the vacuum Einstein equations hold outside some compact region and using them we want to find some additional structure in the metric (5.15). It turns out that assuming that the power series (5.14) exist is very helpful in the following, where we want to discuss the vacuum Einstein equations near infinity. The explicit form of the vacuum Einstein equations

$$R_{ab} = 0 \,, \tag{6.1}$$

where R_{ab} is the Ricci tensor, is given in appendix A in Bondi coordinates. Before considering the general case, we first derive the Schwarzschild metric to illustrate the general idea and steps which will reappear thereafter in the general discussion.

6.1.1 Schwarzschild Higher Dimensions Einstein Equations

To derive the Schwarzschild metric in higher dimensions we use the symmetry and make the ansatz that $\gamma_{AB} = f s_{AB} = s_{AB} + \mathcal{O}(r^{-1/2})$ with f as a scalar function. Plugging this into the rr-component (A.2) we find $f = 1$ and therefore $\gamma_{AB} = s_{AB}$ whence $\dot{\gamma}_{AB} \equiv \partial_r \gamma_{AB} = 0$ and $\gamma'_{AB} \equiv \partial_u \gamma_{AB} = 0$. As a result, all derivatives of γ_{AB} in the other components of the Ricci tensor vanish and the Einstein equations simplify significantly. We start with the R_{rA}-component (A.1), which takes the simple form

$$\frac{d-2}{2} r\beta_A - \frac{d-2}{2} r^2 \dot{\beta}_A - \frac{1}{2} r^3 \ddot{\beta}_A = 0 \,. \tag{6.2}$$

Now, plug in the asymptotic expansion (5.14) for β_A. At order r^k one finds $\beta_A^{(k)} = 0$ if $k \neq 2d-4$ and hence only one term remains such that

$$\beta_A = \beta_A^{(2d-4)} r^{-d+2} \,. \tag{6.3}$$

This can be used to investigate the asymptotic expansions of α. For this, we look at the ru-component (A.4) that now reads

$$-\frac{d-2}{2r}\beta_A\beta^A - \frac{d-2}{2}\beta_A\beta^A + \frac{1}{2}D^A\beta_A + r\left(2\beta^A\dot{\beta}_A + \frac{1}{2}D^A\dot{\beta}_A - (d-2)\dot{\alpha}\right) + r^2\left(-\ddot{\alpha} - \frac{1}{2}\partial_r(\beta^A\dot{\beta}_A)\right) \,. \tag{6.4}$$

Again, plug in the asymptotic expansion for β_A (now simply (6.3)) and α and consider the equation at each order r^k. For $k \neq 2d-4, 4d-8$ the terms with β_A do not contribute and one finds $(2d-6-k)\alpha^{(k)} = 0$ and therefore

$$\alpha^{(k)} = 0 \qquad \text{if } k \neq 2d-4,\ 2d-6,\ 4d-8 \,. \tag{6.5}$$

The two terms in α that are related to β_A are

$$\alpha^{(2d-4)} \propto \mathcal{D}_A \beta^{A(2d-4)} \tag{6.6}$$

and

$$\alpha^{(4d-8)} \propto \beta^{A(2d-4)} \beta_A^{(2d-4)} \,, \tag{6.7}$$

where the prefactors are irrelevant for the discussion and we omit them. From the R_{uu} -component (A.6) we find, at order r^{-4d+10}, that

$$\beta^{A(2d-4)} \beta_A^{(2d-4)} = 0 \,. \tag{6.8}$$

This can be used in the AB-component (A.3), which reduces to

$$\beta_A^{(2d-4)} \beta_B^{(2d-4)} \propto s_{AB} \beta_C^{(2d-4)} \beta^{C(2d-4)} \,, \tag{6.9}$$

to find $\beta_A^{(2d-4)} = 0$. Consequently, all coefficients $\beta_A^{(n)}$ vanish and we have

$$\beta_A = 0 \,. \tag{6.10}$$

Thus, α takes the form

$$\alpha = \alpha^{(2d-6)} r^{-(d-3)} \,. \tag{6.11}$$

Since $\mathcal{D}_A \alpha = 0$ (from R_{uA}) and $\partial_u \alpha = 0$ (from R_{uu}) it follows that $\alpha^{(2d-6)}$ is constant. Summarizing, we have

$$\begin{cases} \alpha^{(2d-6)} r^{-(d-3)} \\ \beta_A = 0 \\ \gamma_{AB} = s_{AB} \end{cases} \tag{6.12}$$

and the metric reads

$$g_{ab}dx^a dx^b = -2\alpha^{(2d-6)}du^2 - 2dudr + r^2 s_{AB}dx^A dx^B \,. \tag{6.13}$$

If we define $M \equiv c/2 := -\alpha^{(2d-6)}$ we arrive at the metric (4.16). That α is constant with respect to x^A and u reflects that the spacetime is static. (6.10) can be viewed a consequence of the rotation symmetry of the ansatz for γ_{AB}.

6.1.2 Recursion Relations from Einstein Equations

In this subsection we will treat the general vacuum Einstein equations in a way very similar to the Schwarzschild case just considered. That is, we plug in the asymptotic expansions (5.14) into $R_{ab} = 0$. The resulting equations are very lengthy but it turns out that it is possible to solve the equations at low orders of r recursively, by looking at each order of r, which yields some restrictions for the coefficients $\alpha^{(n)}, \beta^{(n)}$, and $\gamma_{AB}^{(n)}$. We denote by $\mathcal{R}_{AB}$ the Ricci tensor of γ_{AB}. The results are summarized in the following lemma.

Lemma 6.1. *We assume*

$$\gamma_{AB}^{(0)}\Big|_{u=0} = s_{AB}\,,$$
$$\mathcal{R}_{AB} = \lambda^2(d-3)s_{AB}\,, \text{ and}$$
$$\alpha^{(0)} = \frac{\lambda^2}{2}\,,$$

where $\lambda \in \mathbb{R}$ is a constant. Then we have $\beta_A^{(0)} = 0$ and for $1 \le n \le d-3$,

$$\alpha^{(n)} = 0\,, \tag{6.14}$$

$$\beta_A^{(n)} = 0\,, \tag{6.15}$$

$$\gamma_{AB}^{(n)} = 0\,. \tag{6.16}$$

In addition, we have for $1 \le n \le 2d-5$

$$\beta_A^{(n)} = \frac{n}{2(n+2)(2d-n-4)}\mathcal{D}^B\gamma_{AB}^{(n)} \tag{6.17}$$

and for $1 \le n \le 2d-7$

$$\alpha^{(n)} = \frac{n-2}{n(2d-n-6)}\mathcal{D}^A\beta_A^{(n)}\,. \tag{6.18}$$

Furthermore, if $d > 4$ then for $1 \le n \le 2d-5$ we have

$$\gamma^{(n)} \equiv s^{AB}\gamma_{AB}^{(n)} = 0\,. \tag{6.19}$$

Lastly,

$$\gamma^{(2d-4)} = -\frac{3d-10}{8(d-3)}\gamma^{(d-2)AB}\gamma_{AB}^{(d-2)}\,. \tag{6.20}$$

Our assumptions in this lemma are essentially necessary only to ensure that the spacetime is asymptotically flat and that a Killing spinor exists on Σ, see chapter 5. Thus, the assumptions are not very restrictive. Note that the results in Lemma 6.1 (and also the results found in the rest of *this* chapter) do not change if $\lambda \in \mathbb{C}$.

Proof of Lemma 6.1

The relations in Lemma 6.1 are obtained by plugging the asymptotic expansions (5.14) into the vacuum Einstein equations $R_{ab} = 0$. The complexity of the equations reduces significantly when one looks at the equations at each order of r. Using at each step the results found from lower orders the relations are found.
The first non-trivial equation is found at *order* $r^{3/2}$ where the R_{rr} component gives

$$\gamma^{(1)} = 0. \tag{6.21}$$

At *order* r, R_{ur} yields

$$s^{AB}\gamma'^{(0)}_{AB} = \gamma'^{(0)} = 0 \tag{6.22}$$

while the R_{AB} component gives

$$\gamma'^{(0)}_{AB} = 0. \tag{6.23}$$

We assumed that $\gamma^{(0)}_{AB}\big|_{u=0} = s_{AB}$ but this shows that $\gamma^{(0)}_{AB} = s_{AB}$ for all u, not just for $u = 0$. Using this assumption one finds from $R_{rA} = 0$ that

$$\frac{d-2}{2}\beta^{(0)}_A = 0. \tag{6.24}$$

where $\mathcal{D}^A s_{AB} = 0$ was used which holds since $\mathcal{D}_A$ is the Levi-Civita derivative of s_{AB}. The relation

$$8s^{AB}\gamma^{(2)}_{AB} = \frac{3}{2}\gamma^{(1)AB}\gamma^{(1)}_{AB} \tag{6.25}$$

follows from R_{rr}.

At the next *order*, $r^{1/2}$, using the results from higher orders, leads in $R_{AB} = 0$ to

$$\gamma^{(1)}_{AB} = 0. \tag{6.26}$$

Going back to (6.25) this yields

$$\gamma^{(2)} = 0. \tag{6.27}$$

From the R_{rr} component we find

$$\gamma^{(3)} = 0. \tag{6.28}$$

$$
\begin{tikzcd}
R_{ur} = 0 \arrow[r, "\text{Results from}", "\text{higher order}"'] & \alpha^{(1)} = 0 \arrow[r, "R_{AB}=0"] & \gamma^{(3)}_{AB} = 0 \arrow[d, "R_{rr}=0"] \\
& \beta^{(3)}_A = 0 & 0 = \gamma^{(6)} = \gamma^{(7)} \arrow[l, "R_{rA}=0"']
\end{tikzcd}
$$

Figure 6.1: A sketch of the structure of the argument at order r^0. The argument for other orders of r is similar.

Additionally, we have $\beta^{(1)}_A = 0$ since $R_{rA} = 0$ at the current order reads

$$\frac{2d-7}{8}\beta^{(1)}_A + \frac{1}{4}\left(\mathcal{D}_A\gamma^{(1)} - \mathcal{D}^B\gamma^{(1)}_{AB}\right) = 0. \tag{6.29}$$

If $d = 4$ this is all we can say. Thus, from now on we assume $d > 4$.

At *order* r^0 we get the following equations. From R_{uA} we find

$$\mathcal{D}_A\alpha^{(0)} = 0. \tag{6.30}$$

Assuming that s_{AB} is the metric of a $(d-2)$-dimensional Einstein space we know that $\mathcal{R}_{AB} = \lambda^2(d-3)s_{AB}$ and we assume $\alpha^{(0)} = \frac{\lambda^2}{2}$ which is in accordance with (6.30). With theses two assumptions we find from R_{AB} that

$$(d-4)\gamma'^{(2)}_{AB} = 0. \tag{6.31}$$

To conclude from this that $\gamma^{(2)}_{AB} = 0$ the assumption $d > 4$ is necessary. If $d = 4$ we cannot conclude $\gamma^{(2)}_{AB} = 0$ and it is not possible to say more about the coefficients than done so far. Such a breakdown of the recursion relations will in fact appear for all dimensions and is in fact crucial since it ensures that the series is not trivial and that physically relevant coefficients do not vanish identically. In the R_{rA} component, $\gamma^{(2)}_{AB} = 0$ leads to

$$(d-3)\beta^{(2)}_A + \frac{1}{2}\left(\mathcal{D}_A\gamma^{(2)} - \mathcal{D}^B\gamma^{(2)}_{AB}\right) = (d-3)\beta^{(2)}_A = 0 \tag{6.32}$$

and $R_{rr} = 0$ yields the equations

$$\gamma^{(4)} = \frac{1}{8}\gamma^{(1)AB}\gamma^{(3)}_{AB} = 0 = \gamma^{(5)}. \tag{6.33}$$

Continuing at *order* $r^{-1/2}$, we have to assume $d > 5$ to find new relations. From R_{ur} we find that $\alpha^{(1)} = 0$ which leads in R_{AB} to $\gamma_{AB}^{(3)} = 0$ since we know that $\mathcal{R}_{AB}^{(1)} = 0$ (by assumption). Furthermore, the components R_{rr} and R_{rA} yield $\gamma^{(6)} = \gamma^{(7)} = 0$ and $\beta_A^{(3)} = 0$, respectively. Additionally, one finds, using R_{ur} at order r^{-1} and $r^{-3/2}$, $\beta_A^{(2)} = \beta_A^{(3)} = 0$ implies $\alpha^{(2)} = \alpha^{(3)} = 0$. To summarize, thus far we have found (if $d > 5$):

$$
\begin{aligned}
\alpha^{(0)} &= \frac{\lambda^2}{2} \\
\alpha^{(1)} &= \alpha^{(2)} = \alpha^{(3)} = 0 \\
\beta_A^{(0)} &= \beta_A^{(1)} = \beta_A^{(2)} = \beta_A^{(3)} = 0 \\
\gamma_{AB}^{(0)} &= s_{AB} \\
\gamma_{AB}^{(1)} &= \gamma_{AB}^{(2)} = \gamma_{AB}^{(3)} = 0 \\
\gamma^{(0)} &= \gamma^{(1)} = \gamma^{(2)} = \ldots = \gamma^{(7)} = 0 \\
\mathcal{R}_{AB}^{(0)} &= \lambda^2(d-3)s_{AB} \\
\mathcal{R}_{AB}^{(1)} &= 0
\end{aligned}
$$

We can now continue inductively. At each new *order* $r^{-k/2}$ we have to assume that the dimension is $d > k + 4$. If this is not the case the induction breaks down because we cannot conclude from the R_{AB} component that $\gamma_{AB}^{(k+2)} = 0$. Explicitly, the induction is done as follows. Assume that for some integer $k \geq 2$ we have:

$$
\begin{aligned}
\alpha^{(0)} &= \frac{\lambda^2}{2} \\
\alpha^{(1)} &= \alpha^{(2)} = \ldots = \alpha^{(k)} = 0 \\
\beta^{(0)} &= \beta^{(1)} = \ldots = \beta^{(k+2)} = 0 \\
\gamma_{AB}^{(0)} &= s_{AB} \\
\gamma_{AB}^{(1)} &= \gamma_{AB}^{(2)} = \ldots = \gamma_{AB}^{(k+2)} = 0 \\
\gamma^{(1)} &= \gamma^{(2)} = \ldots = \gamma^{(2k+5)} = 0
\end{aligned}
$$

We also know that $\mathcal{R}_{AB}^{(1)} = \mathcal{R}_{AB}^{(2)} = \ldots = \mathcal{R}_{AB}^{(k)} = 0$. At *order* $r^{(k+1)/2}$ the Einstein equations yield the following relations (assuming $d > k + 4$). The R_{ur} component gives $\alpha^{(k+1)} = 0$ and at the next orders $\alpha^{(k+2)} = \alpha^{(k+3)} = 0$. Thus from R_{AB} we have $(d-k-4)\gamma_{AB}^{(k+3)} = 0$ since $\mathcal{R}_{AB}^{(k+1)} = 0$ and $d > k + 4$. With this R_{rA} and R_{rr} yield $\beta^{(k+3)} = 0$ and $\gamma^{(2k+6)} = \gamma^{(2k+7)} = 0$, respectively.

Therefore, we now have:

$$\begin{aligned}
\alpha^{(0)} &= \frac{\lambda^2}{2} \\
\alpha^{(1)} &= \alpha^{(2)} = \ldots = \alpha^{(k+3)} = 0 \\
\beta^{(0)} &= \beta^{(1)} = \ldots = \beta^{(k+3)} = 0 \\
\gamma^{(0)}_{AB} &= s_{AB} \\
\gamma^{(1)}_{AB} &= \gamma^{(2)}_{AB} = \ldots = \gamma^{(k+3)}_{AB} = 0 \\
\gamma^{(1)} &= \gamma^{(2)} = \ldots = \gamma^{(2k+7)} = 0
\end{aligned}$$

where $k < d - 4$ which completes the inductive step. Thus, we have the equations (6.14), (6.15), and (6.16). If d does not fulfil this inequality we still find some relations between the coefficients from the Einstein equations. Namely, one has at *order* $r^{-n/2}$ from component R_{ur}

$$\alpha^{(n)} \frac{n\,(2d - n - 6)}{4} + \frac{1}{2}\mathcal{D}^A \beta^{(n)}_A - \frac{n}{4}\mathcal{D}^A \beta^{(n)}_A = 0$$

as long as $n \leq 2d - 7$, this is (6.18). For greater n there are many more terms which do not vanish and the equations become very lengthy. $R_{rA} = 0$ simplifies at *order* $r^{-n/2}$ to

$$\beta^{(n+2)}_A \frac{(n+4)(2d - n - 6)}{8} + \frac{n+2}{4}\left(\mathcal{D}_A \gamma^{(n+2)} - \mathcal{D}^B \gamma^{(n+2)}_{AB}\right) = 0$$

if $n \leq 2d - 5$, proofing (6.17) holds. Again, for greater n there are additional terms and no such simple equations are found. It is, of course, still possible to write down the equations but they will not be needed. Furthermore, there is an equation from component R_{rr} for $n = d - 4$,

$$\frac{4n^2 + 12n + 8}{8}\gamma^{(2n+4)} + \frac{-3n^2 - 8n - 4}{16}\gamma^{(n+2)AB}\gamma^{(n+2)}_{AB} = 0\,.$$

This is equal to (6.20) after substituting $n = d - 4$ and hence we have shown all relations in the lemma. □

We have thus found that many components of the asymptotic expansion vanish or are proportional to a total derivative for low orders of r (where "low" is determined by the dimension d). For this, we only needed the assumptions in Lemma 6.1 (Σ is an Einstein manifold and asymptotic flatness), and that the vacuum Einstein equations hold near infinity. The results of Lemma 6.1 will be used in the rest of the thesis to derive a coordinate expression for the Bondi mass and to show its positivity. In particular, they are crucial to show that potentially diverging integrals in fact exist and are well-defined.

6.1.3 Consistency of Asymptotic Expansion

Before we continue with the main thread of this chapter and use the relations just derived to find an expression for the Bondi mass we present an additional argument to justify the asymptotic expansion (5.14) we assumed. We already saw that our choice is consistent with the metric in presence of gravitational radiation. Here, we present briefly a more general argument that can, in principle, be used for every differential equation, where a power series ansatz is used, to check if there are consistency issues. We note that the following argument does not proof that a chosen series is the right ansatz but can only excludes some possibilities. The general idea is as follows. For some function f assume some differential equation, for concreteness take, e.g.,

$$\hat{A}f = r^2 \partial_r^2 f \,, \tag{6.34}$$

where $\hat{A}$ is some operator independent of r, and assume an expansion

$$f \sim \sum_{n\in\mathbb{Z}} f^{(n)} r^{-\lambda} \,, \tag{6.35}$$

where λ is related to n in some way such that $\lambda > 0$ if $n > 0$. Now, plug this series into the differential equation. In our example this yields

$$r^{-\lambda} \hat{A} f^{(n)} = \lambda(\lambda - 1) f^{(n-1)} r^{-\lambda} \,. \tag{6.36}$$

Consider the equation for a given order of r, as was done in the proof of Lemma 6.1 above, in the present example

$$\hat{A} f^{(n)} = \lambda(\lambda - 1) f^{(n-1)} \,. \tag{6.37}$$

For some choices of λ a problem arises as follows. For the sake of argument, take $\lambda = n/2$. Consequently, if we look at (6.37), since the prefactor $n(n/2-1)/2$ does not vanish we see that $f^{(0)}$ is determined by $f^{(-1)}$ (order $r^{1/2}$), $f^{(-1)}$ is then related to $f^{(-2)}$ (order r^1) and so forth. For the series to be non-trivial these coefficients have to be non-zero. However, these parts of the series corresponding to positive powers of r increase with increasing r and thus f does not become small for large r. This is unphysical since we expect that the effect of some source at the origin becomes small for large r. Thus, λ cannot be related to n like $\lambda = n/2$. This way one can check the consistency of a chosen expansion for a given differential equation. We will now use this argument to justify our choice of $\lambda = n/2$ in (5.14). Since the equations, which appear when considering the full Einstein equations, are very lengthy we will restrict ourself in the following to the linearized vacuum Einstein equations, which suffice for our purpose, see also [107]. Additionally, we consider only the case $\beta_A = 0$ and $\alpha = 1/2$. These assumptions are not necessary for the argument, but are used only to keep the expressions shorter. We will look at the R_{AB}-component (A.3), which, with our assumptions, reads

$$0 = \left(\partial_r^2 - 2\partial_u\partial_r\right)\gamma_{AB} - \frac{2d}{r^2}\gamma_{AB} - \frac{6-d}{r}\partial_r\gamma_{AB} + \frac{6-d}{r}\partial_u\gamma_{AB} + s_{AB} \,. \tag{6.38}$$

Now, assume $\gamma_{AB} \sim \sum_n \gamma_{AB}^{(n)} r^{-\lambda}$ with the same conditions on λ as above. Substituting this series yields

$$\sum_n [\lambda(\lambda+1)+2d]\gamma_{AB}^{(n)} r^{-\lambda+1} = \sum_n [d-4-2\lambda]\gamma_{AB}^{\prime(n)} r^{-\lambda} + s_{AB} \tag{6.39}$$

and at order $r^{-\lambda}$ we write this as

$$[\lambda(\lambda+1)+2d]\gamma_{AB}^{(\lambda-1)} = [d-4-2\lambda]\gamma_{AB}^{\prime(\lambda)} + s_{AB}\,. \tag{6.40}$$

By the same reasoning as in the general argument presented above we want the most general λ such that terms which increase with growing distance r (i.e. coefficients corresponding to positive powers of r) are not necessary for a non-trivial series. Let us look at the prefactor on the right-hand side. We see that it vanishes if $\lambda = d/2 - 2$ and, since d is an integer, this means that λ is an integer (half-integer) for even (odd) dimensions. That the prefactor vanishes is needed to avoid the inconsistencies. To see this, take for example at $d = 8$ and $\lambda = n$ an integer. For different values of n we have

$$\begin{aligned}
&\cdots \\
n = -1:&\qquad \gamma_{AB}^{(-2)} \propto \gamma_{AB}^{\prime(-1)} \\
n = 0:&\qquad \gamma_{AB}^{(-1)} \propto \gamma_{AB}^{\prime(0)} \\
n = 1:&\qquad \gamma_{AB}^{(0)} \propto \gamma_{AB}^{\prime(1)} + s_{AB} \\
n = 2:&\qquad \gamma_{AB}^{\prime(1)} = 0 \\
n = 3:&\qquad \gamma_{AB}^{(2)} \propto -\gamma_{AB}^{\prime(3)} \\
n = 4:&\qquad \gamma_{AB}^{(3)} \propto -\gamma_{AB}^{\prime(4)} \\
&\cdots
\end{aligned}$$

where we omitted the unimportant prefactors. For greater/smaller n the list continues in the obvious way without another component vanishing. Notice that due to the vanishing prefactor for $n = 2$ we have $\gamma_{AB}^{(0)} = s_{AB}$ in the $n = 1$ equation and thus $\gamma_{AB}^{(-1)} = 0$ for $n = 0$ and all components corresponding to positive powers of r vanish . Hence, all components which would not decay as $r \to \infty$ are zero by virtue of the prefactor vanishing for $n = 2$. Thus, a different choice for λ, where this does not happen, would have the unwanted coefficients in the series. This shows that the choice of an integer (half-integer) power for r in the asymptotic expansion (5.14) is justified and does not lead to inconsistencies. We now return to the main thread of this chapter and give a definition of the Bondi mass.

6.2 Bondi Mass in Odd Dimensions ≥ 5

We want to find a coordinate expression for the Bondi mass for odd dimensions $d \geq 5$. We start with some geometric definitions (that is without reference to any coordinate system) and show subsequently that the expressions defined exist and have a meaningful form in Bondi coordinates. Let $\Omega = 1/r$. For a sufficiently large distance from the source, and thus for sufficiently small Ω, we define the **Bondi mass density** as

$$\mu_{\tilde{g}} := \frac{1}{8\pi(d-3)}\Omega^{-d+4}\left[\frac{1}{2}\left\langle K_{\tilde{g}} - \tilde{g}, \mathrm{Hess}_{\tilde{g}} u\right\rangle_{\tilde{g}} - \Omega^{-1}C_{\tilde{g}}\left(\frac{\partial}{\partial\Omega}, \mathrm{grad}_{\tilde{g}}\Omega, \frac{\partial}{\partial\Omega}, \mathrm{grad}_{\tilde{g}}\Omega\right)\right] \quad (6.41)$$

where

$$K_g := \frac{2}{d-2}\mathrm{Ric}_g - \frac{1}{(d-1)(d-2)}g \cdot \mathrm{Scal}_g \quad (6.42)$$

is the Schouten tensor. This expression was derived for even-dimensional spacetimes in [17] using the Hamiltonian formalism [56]. See also Appendix C for a summary of the derivation. Near null infinity the $(d-2)$−dimensional surface of constant r and u is denoted by $\Sigma(u, r)$. For a given asymptotically null surface one can now define the Bondi mass as an surface integral over the Bondi mass density at null infinity, i.e., the **Bondi mass** of $\Sigma(0, \infty)$ is defined as

$$m_\Sigma = \lim_{r\to\infty}\int_{\Sigma(0,r)} \mu_{\tilde{g}} dS_{\tilde{g}}\,, \quad (6.43)$$

where $dS_{\tilde{g}}$ is the induced integration element on $\Sigma(u, r)$. To facilitate the interpretation of the explicit Bondi mass formula, which we derive below, the following definition [17, 19, 36] is advantageous. The **Bondi news tensor** is defined as

$$N = \lim_{r\to\infty}\left[r^{d/2-2}\left(K_{\tilde{g}} - \tilde{g}\right)\right]\,. \quad (6.44)$$

The Bondi news will turn out to be related to the mass changing over time which is due to gravitational radiation. It is not obvious at all that the two expressions (6.43) and (6.44) are well defined and that the limits exist. To see that this is the case we try to find an expression of these quantities in Bondi coordinates just like in $d = 4$. The relations in Lemma 6.1 can be used to show that the definitions 6.43 and 6.44 are meaningful and, furthermore, a relatively simple expression in terms of Bondi coordinates can be found. This central result of this chapter is

Theorem 6.2. *Assume that Lemma 6.1 holds. Then, the limit in* (6.43) *exists and in coordinates the Bondi mass of* Σ *is given by*

$$m_\Sigma = \frac{(d-2)}{8\pi}\int_\Sigma\left(\frac{1}{8(d-3)}\gamma^{(d-2)AB}\gamma'^{(d-2)}_{AB} - \alpha^{(2d-6)}\right)\sqrt{s}d^{d-2}x\,. \quad (6.45)$$

The limit in (6.44) *exists, too, and in coordinates the only non-trivial component of the Bondi news is*

$$N_{AB} = -\gamma'^{(d-2)}_{AB} \ . \tag{6.46}$$

We have the ***mass-loss formula***

$$\frac{d}{du} m_{\Sigma(u,0)}|_{u=0} = -\frac{1}{32\pi} \int_{\Sigma} \langle N, N \rangle_{\tilde{g}} \, dS_{\tilde{g}} \leq 0 \ . \tag{6.47}$$

Eq. (6.47) shows that the change of the Bondi mass with time is determined by the Bondi News tensor which is related to gravitational waves. Thus, one may view this as saying that the mass of a system changes by emission of gravitational radiation, as might be expected. Therefore, the first term in (6.45) corresponds to gravitational waves. The second term (containing $\alpha^{(2d-6)}$) can be shown to be equal to the parameter M in the usual Schwarzschild (see section 6.1.1) or Myers-Perry [108, 109] metrics. To see this one has to look at the uu-component of the metric in Bondi coordinates. In the Newtonian limit, this is also the term which corresponds to the mass in the Newtonian potential.

Proof of Theorem 6.2

The theorem, and the proof, consist of three parts. First, we proof that the limit in (6.44) exists and that the Bondi News in coordinates is (6.46). Thereafter, the same is shown for the Bondi mass (6.43). Finally, the mass-loss formula (6.47) is derived. Note that we drop the factor of $1/8\pi$ appearing in (6.41) in the following proof.

Part 1. We write $\tilde{K}_{ij}$ and K_{ij} for the Schouten tensor corresponding to the unphysical and physical metric, respectively. We have that the conformal transformation of the Schouten tensor is

$$\begin{aligned}
\frac{1}{2}\tilde{K}_{ij} &= \frac{1}{2}K_{ij} - \nabla_i(\Omega^{-1}\partial_j\Omega) + \Omega^{-1}(\partial_i\Omega)\Omega^{-1}(\partial_j\Omega) - \frac{1}{2}g_{ij}\Omega^{-1}(\partial_k\Omega)\Omega^{-1}\partial^k\Omega \\
&= \frac{1}{2}K_{ij} + \Omega^{-2}(\partial_i\Omega)\partial_j\Omega - \nabla_i(\Omega^{-1}\partial_j\Omega) - \frac{1}{2\Omega^2}g_{ij}g^{\Omega\Omega} \\
&= \Omega^{-2}(\partial_i\Omega)(\partial_j\Omega) - \nabla_i(\Omega^{-1}\partial_j\Omega) - \frac{1}{2\Omega^2}g_{ij}g^{\Omega\Omega} \ .
\end{aligned}$$

In the last equality it was used that the vacuum Einstein equations $R_{ab} = 0$ imply that $K_{ij} = 0$. For the AB-component of the Bondi news tensor we need only the AB-component of the Schouten tensor,

$$\begin{aligned}
\frac{1}{2}\tilde{K}_{AB} &= -\nabla_A(\Omega^{-1}\partial_B\Omega) - \frac{1}{2\Omega^2}g_{AB}g^{\Omega\Omega} = \Gamma^C_{AB}\Omega^{-1}\partial_C\Omega - \frac{1}{2\Omega^2}g_{AB}g^{\Omega\Omega} \\
&= \Gamma^{\Omega}_{AB}\Omega^{-1} - \frac{1}{2\Omega^2}g_{AB}g^{\Omega\Omega} \ .
\end{aligned} \tag{6.48}$$

With $\Omega = 1/r$ we know that

$$\frac{r^2}{2} g_{AB} g^{rr} = \frac{r^2}{2} r^2 \gamma_{AB} r^{-4}(2\alpha + \beta^C \beta_C) = \frac{1}{2}\gamma_{AB}(2\alpha + \beta^C \beta_C).$$

The relevant Christoffel symbol is found to be

$$\Gamma^r_{AB} = -\frac{1}{r^2}\alpha\dot{\gamma}_{AB} + \frac{2}{r}\alpha\gamma_{AB} - \frac{1}{2r^2}\beta^C\beta_C\dot{\gamma}_{AB} + \frac{1}{r}\beta^C\beta_C\gamma_{AB} - \frac{1}{r}\partial_{(A}\beta_{B)} - \frac{\gamma'_{AB}}{2}.$$

Plugging this into (6.48) yields

$$\begin{aligned}
\frac{1}{2}\tilde{K}_{AB} &= -\frac{1}{r}\alpha\dot{\gamma}_{AB} - \partial_{(A}\beta_{B)} - \frac{r}{2}\gamma'_{AB} - \frac{1}{2r}\beta^C\beta_C\dot{\gamma}_{AB} + \gamma_{AB}(\beta_C\beta^C + 2\alpha) - \frac{1}{2}\gamma_{AB}(2\alpha + \beta^C\beta_C) \\
&= -\partial_{(A}\beta_{B)} - \frac{r}{2}\gamma'_{AB} + \frac{1}{2}\gamma_{AB}(\beta^C\beta_C + 2\alpha) - \frac{1}{2r}\dot{\gamma}_{AB}(2\alpha + \beta_C\beta^C) \\
&= -\partial_{(A}\beta_{B)} - \frac{r}{2}\gamma'_{AB} + \frac{1}{2}\left(\gamma_{AB} - \frac{1}{r}\dot{\gamma}_{AB}\right)\left(\beta^C\beta_C + 2\alpha\right).
\end{aligned}$$

Hence, we have for the AB-component of the news tensor

$$N_{AB} = \lim_{r\to\infty}\left[r^{d/2-2}\left\{-2\partial_{(A}\beta_{B)} - r\gamma'_{AB} + \left(\gamma_{AB} - \frac{1}{r}\dot{\gamma}_{AB}\right)\left(\beta^C\beta_C + 2\alpha\right) - \gamma_{AB}\right\}\right].$$

Substituting the asymptotic expansions and using the results from Lemma 6.1 we find

$$\begin{aligned}
N_{AB} &= \lim_{r\to\infty} r^{d/2-2}\left[s_{AB} - r^{-d/2+2}\gamma'^{(d-2)}_{AB} - s_{AB} + O(r^{-d/2+1})\right] \\
&= -\gamma'^{(d-2)}_{AB}
\end{aligned} \tag{6.49}$$

In particular, one sees that all terms which are divergent in the limit $r \to \infty$ are equal to zero or cancel and thus the limit exists. All other components of N_{ab} are found to be zero and (6.46) holds.

Part 2. We start the treatment of the Bondi mass density by rewriting the definition

$$\mu_{\tilde{g}} = \frac{1}{d-3}\Omega^{-d+4}\left[\underbrace{\frac{1}{2}(\tilde{K}^{ab} - \tilde{g}^{ab})\tilde{\nabla}_a\tilde{\nabla}_b u}_{(I)} - \frac{1}{\Omega}\underbrace{\tilde{C}^{abcd}(\nabla_a u)(\nabla_b\Omega)(\nabla_c u)\nabla_d\Omega}_{(II)}\right] \tag{6.50}$$

in coordinates which is done analogously to the treatment of Bondi news tensor above. That is, explicit expressions of the Schouten and Weyl tensor in our coordinate system have to be found. Then, the asymptotic expansion and the results from Lemma 6.1 are used to simply the expressions and to show that the limit exists. The two terms in (6.50), (I) and (II), will be treated separately to simplify the expressions appearing in the following calculation. As will be argued below, they also correspond to two distinct physical processes, so the split is rather natural. We know that

$$\tilde{\nabla}_a \tilde{\nabla}_b u = \partial_a \partial_b u - \tilde{\Gamma}^c_{ab} \partial_c u = -\tilde{\Gamma}^u_{ab} = \frac{1}{2} \partial_\Omega \tilde{g}_{ab}$$

and then we can rewrite (I) as

$$\begin{aligned}(\mathrm{I}) &= -\frac{1}{4}\left([\tilde{K}_{AB} - \tilde{g}_{AB}]\partial_\Omega \tilde{g}^{AB} + [\tilde{K}_{u\Omega} - \tilde{g}_{u\Omega}]\partial_\Omega \tilde{g}^{u\Omega} + [\tilde{K}_{\Omega A} - \tilde{g}_{\Omega A}]\partial_\Omega \tilde{g}^{\Omega A}\right) \\ &= -\frac{1}{4}\left([\tilde{K}_{AB} - \gamma_{AB}]\partial_\Omega \gamma^{AB} + \tilde{K}_{\Omega A}\partial_\Omega \tilde{g}^{\Omega A}\right) . \end{aligned} \tag{6.51}$$

The term $\tilde{K}_{AB} - \gamma_{AB}$ appeared already in the treatment of the Bondi news above and the other relevant component of the Schouten tensor is found to be

$$\tilde{K}_{\Omega A} = 2\Gamma^\Omega_{\Omega A}\Omega^{-1} = -\partial_\Omega \beta_A - \frac{1}{\Omega}\beta_A + \Omega \beta^B \partial_\Omega \gamma_{AB} .$$

Now, the asymptotic expansion (5.14) and the relations from Lemma 6.1 are used and after a short calculation one finds

$$(I) = \frac{d-2}{8} r^{4-d} \gamma'^{(d-2)}_{AB} \gamma^{AB(d-2)} + O(r^{3-d}) . \tag{6.52}$$

We proceed similarly with term (II). We first need to compute the relevant components of the Weyl tensor,

$$\begin{aligned}\tilde{C}^{u\Omega u\Omega} &= \tilde{g}^{au}\tilde{g}^{b\Omega}\tilde{g}^{cu}\tilde{g}^{u\Omega}\tilde{C}_{abcd} = \tilde{g}^{\Omega u}\tilde{g}^{b\Omega}\tilde{g}^{\Omega u}\tilde{g}^{u\Omega}\tilde{C}_{\Omega b\Omega d} \\ &= \tilde{g}^{\Omega u}\tilde{g}^{u\Omega}\tilde{g}^{\Omega u}\left[\tilde{g}^{\Omega\Omega}\tilde{C}_{\Omega u\Omega\Omega} + \tilde{g}^{u\Omega}\tilde{C}_{\Omega u\Omega u} + \tilde{g}^{A\Omega}\tilde{C}_{\Omega u\Omega A}\right] \\ &+ \tilde{g}^{\Omega u}\tilde{g}^{A\Omega}\tilde{g}^{\Omega u}\left[\tilde{g}^{\Omega\Omega}\tilde{C}_{\Omega A\Omega\Omega} + \tilde{g}^{u\Omega}\tilde{C}_{\Omega A\Omega u} + \tilde{g}^{B\Omega}\tilde{C}_{\Omega A\Omega B}\right] \\ &= \tilde{C}_{\Omega u\Omega u} + 2\Omega\beta^A\tilde{C}_{\Omega u\Omega A} + \Omega^2\beta^A\beta^B\tilde{C}_{\Omega A\Omega B} ,\end{aligned} \tag{6.53}$$

where the symmetry $\tilde{C}_{abcd} = \tilde{C}_{cdab}$ was used. Two of these three components of the Weyl tensor have to be computed. The last term will not contribute in the end due to the factor $\beta^A\beta^B$ which is of such an order in r that it will vanish in the limit $r \to \infty$ and we therefore omit this term in the following. From the definition of the Weyl tensor we have

$$\begin{aligned}\tilde{C}_{\Omega u\Omega u} &= \tilde{R}_{\Omega u\Omega u} + \frac{1}{d-2}(2\tilde{R}_{\Omega u}\tilde{g}_{u\Omega} - \tilde{R}_{\Omega\Omega}\tilde{g}_{uu}) - \frac{1}{(d-1)(d-2)}\tilde{R}\tilde{g}^2_{\Omega u} \\ &= \tilde{R}_{\Omega u\Omega u} + \tilde{K}_{\Omega u}\tilde{g}_{u\Omega} - \frac{1}{d-2}\tilde{R}_{\Omega\Omega}\tilde{g}_{uu} \\ &= \tilde{R}_{\Omega u\Omega u} + \tilde{K}_{\Omega u} - \frac{1}{d-2}\tilde{R}_{\Omega\Omega}\tilde{g}_{uu}\end{aligned}$$

The unphysical Riemann tensor can be found from the usual definition in terms of Christoffel symbols

$$\begin{aligned}
\tilde{R}_{\Omega u \Omega u} &= \tilde{g}_{\Omega\alpha}\tilde{R}^{\alpha}_{u\Omega u} = \tilde{g}_{\Omega u}\tilde{R}^{u}_{u\Omega u} = \tilde{R}^{u}_{u\Omega u} \\
&= \partial_\Omega \tilde{\Gamma}^u_{uu} - \partial_u \tilde{\Gamma}^u_{\Omega u} + \tilde{\Gamma}^u_{\Omega e}\tilde{\Gamma}^e_{uu} - \tilde{\Gamma}^u_{ue}\tilde{\Gamma}^e_{\Omega u} \\
&= \partial_\Omega \tilde{\Gamma}^u_{uu} - \partial_u \tilde{\Gamma}^u_{\Omega u} + \tilde{\Gamma}^u_{\Omega\Omega}\tilde{\Gamma}^\Omega_{uu} + \tilde{\Gamma}^u_{\Omega A}\tilde{\Gamma}^A_{uu} - \tilde{\Gamma}^u_{u\Omega}\tilde{\Gamma}^\Omega_{\Omega u} - \tilde{\Gamma}^u_{uA}\tilde{\Gamma}^A_{\Omega u} \\
&= \partial_\Omega[2\Omega\alpha + \Omega^2\partial_\Omega\alpha] + \frac{1}{4}(\beta^A\beta_A + \Omega^2\gamma^{AB}(\partial_\Omega\beta_B)(\partial_\Omega\beta_A) + 2\Omega\beta^A\partial_\Omega\beta_A) \\
&= 2\alpha + 4\Omega\partial_\Omega\alpha + \Omega^2\partial^2_\Omega\alpha + \frac{1}{4}(\beta^A\beta_A + \Omega^2\gamma^{AB}(\partial_\Omega\beta_B)(\partial_\Omega\beta_A) + 2\Omega\beta^A\partial_\Omega\beta_A)\,,
\end{aligned}$$

where the Christoffel symbols

$$\begin{aligned}
&\tilde{\Gamma}^u_{uu} = 2\Omega\alpha + \Omega^2\partial_\Omega\alpha \qquad \tilde{\Gamma}^u_{\Omega u} = 0 \qquad \tilde{\Gamma}^u_{\Omega A} = 0 \\
&\tilde{\Gamma}^A_{\Omega u} = -\frac{1}{2}(\beta^A + \Omega\gamma^{AB}\partial_\Omega\beta_B) \qquad \tilde{\Gamma}^u_{\Omega\Omega} = 0 \qquad \tilde{\Gamma}^u_{uA} = \frac{1}{2}(\beta_A + \Omega\partial_\Omega\beta_A)
\end{aligned}$$

were used. Together with

$$\tilde{K}_{\Omega u} = -2\Omega\partial_\Omega\alpha - \Omega\beta^A\partial_\Omega\beta_A - 2\alpha$$

and

$$\tilde{R}_{\Omega\Omega} = -\frac{1}{2}\gamma^{AB}\partial^2_\Omega\gamma_{AB} + \frac{1}{4}\gamma^{CA}\gamma^{DB}(\partial_\Omega\gamma_{AB})(\partial_\Omega\gamma_{CD})$$

we finally arrive at

$$\begin{aligned}
\tilde{C}_{\Omega u\Omega u} &= 2\Omega\partial_\Omega\alpha + \Omega^2\partial^2_\Omega\alpha + \frac{1}{4}\beta^A\beta_A + \frac{\Omega^2}{4}\gamma^{AB}(\partial_\Omega\beta_B)(\partial_\Omega\beta_A) - \frac{\Omega}{2}\beta^A\partial_\Omega\beta_A \\
&\quad + \frac{\Omega^2\alpha}{d-2}\left(\gamma^{AB}\partial^2_\Omega\gamma_{AB} - \frac{1}{2}\gamma^{CA}\gamma^{DB}(\partial_\Omega\gamma_{AB})(\partial_\Omega\gamma_{CD})\right)\,.
\end{aligned}$$

Substituting the asymptotic expansions, with $\Omega = 1/r$, yields

$$\begin{aligned}
C_{ruru} &= \sum_n 2\cdot\frac{n}{2}\cdot\frac{\alpha^{(n)}}{r^{n/2}} + 2\left(\frac{-n}{2}\right)\cdot\frac{\alpha^{(n)}}{r^{n/2}} + \frac{n}{2}\cdot\frac{n+2}{2}\cdot\frac{\alpha^{(n)}}{r^{n/2}} + O(r^{-(d-2)}) \\
&= \sum_n \frac{\alpha^{(n)}}{r^{n/2}}\left(\frac{n}{2} + \frac{n^2}{4}\right) + O(r^{-(d-2)})\,.
\end{aligned} \tag{6.54}$$

For $\tilde{C}_{\Omega u\Omega A}$ the relevant components of the Riemann and Ricci tensor have to be computed. Since the calculation is very similar to the one for $\tilde{C}_{\Omega u\Omega u}$ we skip the details and just state the result:

$$\tilde{C}_{\Omega u\Omega A} = \frac{1}{2}\left(2\partial_\Omega\beta_A + \Omega\partial^2_\Omega\beta_A\right) + \frac{1}{4}(\partial_\Omega\gamma_{AB})\left(\Omega\gamma^{BC}\partial_\Omega\beta_C - \beta^B\right) - \frac{1}{d-2}\left(\Omega\beta_A\tilde{R}_{\Omega\Omega} + \tilde{R}_{\Omega A}\right)\,.$$

Therefore, the term $2r\beta^A\tilde{C}_{rurA}$ appearing in (6.53) is of order $O(r^{-(d-2)})$ and will not contribute in the end. We are thus left with only one term in (II) which will be relevant. We summarize what we have found so far:

$$
\begin{aligned}
(I) &= \frac{d-2}{8} r^{4-d} \gamma'^{(d-2)}_{AB} \gamma^{AB(d-2)} + O(r^{3-d}) \\
(II) &= \sum_n \frac{\alpha^{(n)}}{r^{n/2}} \left(\frac{n}{2} + \frac{n^2}{4} \right) + O(r^{2-d}) \,.
\end{aligned}
$$

Plugging both terms into (6.43) leads to

$$
\mu_{\tilde{g}} = \frac{1}{d-3} r^{d-4} \left[\frac{d-2}{8} r^{4-d} \gamma'^{(d-2)}_{AB} \gamma^{AB(d-2)} - r \sum_n \frac{\alpha^{(n)}}{r^{n/2}} \left(\frac{n}{2} + \frac{n^2}{4} \right) + O(r^{3-d}) \right] . \tag{6.55}
$$

With this expression at hand we can now go back to (6.43) and investigate whether taking the limit leads to a meaningful expression for the Bondi mass. It is easy to see that the part from (I) is well defined in the limit. To see that the same holds for (II) a closer inspection is necessary. One sees that all terms with $n > 2d-6$ are irrelevant since they vanish due to the limit. All terms $n < 2d-6$ diverge when taking the limit, but using the relation (6.18), i.e. $\alpha^{(n)} \propto \mathcal{D}^A \beta^{(n)}_A$, it is possible to write all of these terms as total divergencies which vanish under the integral. To see this recall that, in general, for a Levi-Civita derivative ∇ of some diagonal metric h we have $\nabla_a v_b = \partial_a v_b - \Gamma^c_{ab} v_c$ and $d \star v = \nabla_a v^a \sqrt{|h|} d^n x$. For the metric s_{AB} with covariant derivative $\mathcal{D}_A$ this leads to

$$
\begin{aligned}
\mathcal{D}^A \beta_A &= \partial^A \beta_A - s^{AB} \Gamma^C_{AB} \beta_C \\
&= \partial^A \beta_A - \beta_C s^{CE} \partial^A s_{AE} + \frac{1}{2} \beta_C s^{AB} \partial^C s_{AB}
\end{aligned}
$$

and for the metric γ_{AB}, with covariant derivative D^A,

$$
\begin{aligned}
D^A \beta_A &= \partial^A \beta_A - \gamma^{AB} \bar{\Gamma}^C_{AB} \beta_C \\
&= \partial^A \beta_A - \beta_C \gamma^{CE} \partial^A \gamma_{AE} + \frac{1}{2} \beta_C \gamma^{AB} \partial^C \gamma_{AB}
\end{aligned}
$$

Since, by Lemma 6.1,

$$
\gamma_{AB} = s_{AB} + \sum_{k \geq d-2} \frac{\gamma^{(k)}_{AB}}{r^{k/2}}
$$

we have $D^A \beta^{(k)}_A = \mathcal{D}^A \beta^{(k)}_A$ for $0 \leq k \leq 2d-5$. Therefore, the terms

$$
\int_{\Sigma(r,0)} \mathcal{D}^A \beta^{(k)}_A \sqrt{\gamma} d^{d-2} x \,,
$$

with $dS_{\tilde{g}} = \sqrt{\gamma} d^{d-2}x$ plugged in, vanish. Hence, the only term which does not vanish is the one where $n = 2d - 6$.
Thus, we finally arrive at the expression for the Bondi mass

$$m_\Sigma = (d-2) \int \left[\frac{1}{8(d-3)} \gamma'^{(d-2)}_{AB} \gamma^{AB(d-2)} - \alpha^{(2d-6)} \right] \sqrt{s} d^{d-2}x \tag{6.56}$$

where the measure is found by using that $\gamma^{(n)} = 0$ for $1 \leq n \leq 2d-5$ and hence, after taking the limit $r \to \infty$,

$$dS_{\tilde{g}} = \sqrt{\gamma} d^{d-2}x \sim \sqrt{s} d^{d-2}x \ .$$

In particular, this shows that the limit in the definition of m_Σ exists.

Part 3. To proof the mass-loss formula, we look at the uu- and rr-components of the Einstein equations. Together, they yield an expression which connects the integrand in the Bondi mass to the Bondi news tensor. This yields the desired expression for the mass loss.
We start by taking the derivative of m_Σ:

$$\frac{d}{du} m_\Sigma = (d-2) \int_\Sigma \left[\frac{1}{8(d-3)} \left(\gamma'^{(d-2)AB} \gamma'^{(d-2)}_{AB} + \gamma^{(d-2)AB} \gamma''^{(d-2)}_{AB} \right) - \alpha'^{(2d-6)} \right] \sqrt{s} d^{d-2}x \ . \tag{6.57}$$

To bring this into the claimed form we use the Einstein equations. At order r^{-N}, where $N = d-2$, we have

$$R_{uu} = -\frac{1}{2} \left(s^{AB} \gamma''^{(2N)}_{AB} + \gamma^{(N)AB} \gamma''^{(N)}_{AB} \right) - \frac{1}{4} \gamma'^{(N)AB} \gamma'^{(N)}_{AB} + (d-2)\alpha'^{(2N-2)} + \mathcal{D}_A w^A = 0 \tag{6.58}$$

where $w^A = r^{-2} \mathcal{D}^A \alpha - r^{-1} \beta'^A$. From (6.20) we have

$$\gamma^{(2N)} = \frac{10-3d}{8(d-3)} \gamma^{(N)AB} \gamma^{(N)}_{AB}$$

and thus

$$\partial_u^2 \gamma^{(2N)} \equiv s^{AB} \gamma''^{(2N)}_{AB} = \frac{10-3d}{4(d-3)} \left(\gamma^{(N)AB} \gamma''^{(N)}_{AB} + \gamma'^{(N)AB} \gamma'^{(N)}_{AB} \right) \tag{6.59}$$

Substituting (6.59) into (6.58) yields

$$\begin{aligned} &\frac{3d-10}{4(d-3)} \left(\gamma^{(N)AB} \gamma''^{(N)}_{AB} + \gamma'^{(N)AB} \gamma'^{(N)}_{AB} \right) - \gamma^{(N)AB} \gamma''^{(N)}_{AB} - \frac{1}{2} \gamma'^{(N)AB} \gamma'^{(N)}_{AB} + 2(d-2)\alpha'^{(2N-2)} \\ &= \frac{3d-10-4(d-3)}{4(d-3)} \gamma^{(N)AB} \gamma''^{(N)}_{AB} + \frac{3d-10-2(d-3)}{4(d-3)} \gamma'^{(N)AB} \gamma'^{(N)}_{AB} + 2(d-2)\alpha'^{(2N-2)} = 0 \end{aligned}$$

We drop the total differential $\mathcal{D}^A w_A$, which will vanish under the integral, in the following computation for simplicity. We arrive at

$$(d-4) \gamma'^{(N)AB} \gamma'^{(N)}_{AB} = (d-2) \gamma^{(N)AB} \gamma''^{(N)}_{AB} - 8(d-3)(d-2)\alpha'^{(2N-2)} \ ,$$

Adding $(d-2)\gamma'^{(N)AB}\gamma'^{(N)}_{AB}$ on both sides and dividing by $8(d-3)$ yields

$$\frac{(d-3)}{4(d-3)}\gamma'^{(N)AB}\gamma'^{(N)}_{AB} = \frac{(d-2)}{8(d-3)}\gamma^{(N)AB}\gamma''^{(N)}_{AB} + \frac{(d-2)}{8(d-3)}\gamma'^{(N)AB}\gamma'^{(N)}_{AB} - (d-2)\alpha'^{(2N-2)} .$$

We can plug this into (6.57) and find

$$\frac{d}{du}m_\Sigma = \int_\Sigma \frac{1}{4}\gamma'^{(N)AB}\gamma'^{(N)}_{AB}\sqrt{s}d^{d-2}x = -\frac{1}{4}\int_\Sigma N^{AB}N_{AB}\sqrt{s}d^{d-2}x \leq 0 .$$

In the last step we raised the indices of $N_{AB} = -\gamma'^{(d-2)}_{AB}$ finding

$$N^{AB} = s^{AC}s^{BD}N_{CD} = -s^{AC}s^{BD}\gamma'^{(d-2)}_{CD} = \gamma'^{(d-2)AB} \tag{6.60}$$

where the general fact

$$\begin{aligned}
\partial_a\gamma^{AB} &= \partial_a\left(\gamma^{AC}\gamma^{BD}\gamma_{CD}\right) = \partial_a\left(\gamma^{AC}\right)\gamma^{BD}\gamma_{CD} + \gamma^{AC}\partial_a\left(\gamma^{BD}\right)\gamma_{CD} + \gamma^{AC}\gamma^{BD}\partial_a\left(\gamma_{CD}\right) \\
&= 2\partial_a\left(\gamma^{AC}\right)\underbrace{\gamma^{BD}\gamma_{CD}}_{\delta^B_C} + \gamma^{AC}\gamma^{BD}\partial_a\left(\gamma_{CD}\right) = 2\partial_a\left(\gamma^{AB}\right) + \gamma^{AC}\gamma^{BD}\partial_a\left(\gamma_{CD}\right) \\
&\Rightarrow \partial_a\gamma^{AB} = -\gamma^{AC}\gamma^{BD}\partial_a\gamma_{CD}
\end{aligned}$$

was used. This shows that the change in mass is always negative so the mass can only decrease or stay constant. This change is characterized by N_{AB} and, in particular, $dm_\Sigma/du = 0$ iff $N_{AB} = 0$. □

This concludes the proof of the main results of this chapter. In this chapter, we investigated the asymptotic expansion of the metric coefficients using the vacuum Einstein equations. The main results are listed in Lemma 6.1. Using these results we found a generalization of the four dimensional "Bondi formulas", discussed in section 4.2, in odd dimensions, namely a coordinate expression for the Bondi mass, (6.2), and an expression for the change of mass over time (6.47). We will discuss these results further in section 8.1, where we also compare them to previous results found by others. We now return to the issue of positivity and discuss in the next chapter how a proof of $m_\Sigma \geq 0$ can be established.

CHAPTER 7

Positivity of Bondi Mass

In this chapter we proof that the Bondi mass is non-negative, that is the Bondi mass is zero for Minkowski spacetime and positive otherwise. This shows that there is a stable ground state and that systems in higher dimensional general relativity are not inherently unstable. We follow the idea of [14], who proofed positivity in $d = 4$, and [19, 110], who proofed positivity in higher even dimensions. This method of proof requires that spinors exist on the manifold. Thus, we start by stating the assumptions necessary to define spinors. Then, we derive explicit expressions for gamma matrices. This is then used to show that the Bondi mass is non-negative. More precisely, we want to proof the following statement.

Assume that there is a Witten spinor on the hypersurface $\mathcal{H} = \left\{u = \frac{1}{2r}\right\}$ *near infinity, that there is a Killing spinor on* (Σ, s) *and that the results from Lemma 6.1 hold. Then,* $m_\Sigma \geq 0$.

7.1 Spin structure, Spinors, Tetrad and Gamma Matrices

In this section we adapt the general definitions from chapter 3 to our coordinate system to facilitate and enable calculations in the subsequent sections. First, we state our assumptions about the spin structure on $\mathscr{M}$. Then, we choose an explicit tetrad system and define gamma matrices in this system. Thirdly, an explicit formula for the spin connection is derived and, lastly, the Witten equation is stated.

7.1.1 Spin Manifold

Let $\mathscr{M}$ be a manifold with fixed spin structure. Consider the Clifford algebra $Cl_{d-1,1}(q, \mathbb{R}^n)$ where $q = -x_0^2 + x_1^2 + \ldots + x_{d-1}^2$. There is an associated Clifford bundle $Cl(T\mathscr{M})$ and locally, at point $p \in \mathscr{M}$, $Cl(T_p\mathscr{M})$ is generated by the identity and elements $\{e_{\mu_1}, \ldots, e_{\mu_n}\}$ subject to the relations

$$e_\nu \cdot e_\mu + e_\mu \cdot e_\mu = 2g(e_\mu, e_\nu)I\,, \tag{7.1}$$

where e_μ, $\mu = 0, \ldots, d-1$, is a basis of $T_p\mathscr{M}$ and I is the identity. The complexification of $Cl(T\mathscr{M})$ has its fundamental representation on a complex vector space and associated to this is a complex vector bundle $\mathscr{S}(\mathscr{M})$. Spinors are smooth sections of $\mathscr{S}(\mathscr{M})$. Recall that on the complexified Clifford algebra there is a positive definite hermitian inner product $\langle\,,\,\rangle$ for spinors, see section 3.3, such that

$$\langle\Psi, \Phi\rangle := \bar{\Psi}\Phi\,, \tag{7.2}$$

where $\Phi \in \Gamma(\mathscr{S})$ and $\bar{\Psi} : \mathscr{S} \to \mathbb{C}$. With this we can define the 2-form

$$Q(X,Y) \coloneqq \Re\left[\bar{\psi} Y \cdot \nabla_X \psi - \bar{\psi} X \cdot \nabla_Y \psi\right] \tag{7.3}$$

on $\mathscr{M}$, where $X, Y \in T\mathscr{M}$, $\Re$ is the real part and ∇ the spin connection.

7.1.2 Tetrad

Note that the unphysical Bondi metric is

$$\tilde{g}_{ab} dx^a dx^b = -2\Omega^2 \alpha du^2 - 2du d\Omega - 2\Omega\beta_A du dx^A + \gamma_{AB} dx^A dx^B . \tag{7.4}$$

We want to define a tetrad with respect to $\tilde{g}$. That is we have a orthonormal basis of smooth vector fields $\{e^\mu_a\}$ where lower case latin indices are spacetime indices taking values $\{r, u, x^A\}$ and lower case greek indices are tetrad indices taking values $\{+, -, I\}$[1]. The **tetrad** is defined by

$$\tilde{g}_{ab} \tilde{e}^{\mu a} \tilde{e}^{\nu b} = \tilde{e}^\mu_a \tilde{e}^{\nu a} = \lambda^{\mu\nu} \tag{7.5}$$

or, equivalently,

$$\tilde{g}_{ab} = \lambda_{\mu\nu} \tilde{e}^\nu_a \tilde{e}^\mu_b , \tag{7.6}$$

where

$$(\lambda_{\mu\nu}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \delta_{IJ} \end{pmatrix} . \tag{7.7}$$

Therefore, we have the following conditions for $\{\tilde{e}^\mu_a\}$ which follow directly from (7.5):

$$\begin{aligned} &\tilde{g}(\tilde{e}^+, \tilde{e}^-) = 1 \\ &\tilde{g}(\tilde{e}^+, \tilde{e}^+) = \tilde{g}(\tilde{e}^-, \tilde{e}^-) = \tilde{g}(\tilde{e}^+, \tilde{e}^I) = \tilde{g}(\tilde{e}^-, \tilde{e}^I) = 0 \\ &\tilde{g}(\tilde{e}^I, \tilde{e}^J) = \delta^{IJ} \end{aligned} \tag{7.8}$$

As can easily be checked a choice consistent with these conditions is

$$\begin{cases} \tilde{e}^{+a} & = \partial^a_\Omega \\ \tilde{e}^{-a} & = \Omega^2 \alpha \partial^a_\Omega + \partial^a_u \\ \tilde{e}^{Ia} & = \Omega \beta_A l^{IA} \partial^a_\Omega + l^{IA} \partial^a_A \end{cases} \tag{7.9}$$

[1]This unusual choice of indices will be explained in subsection (7.1.4).

where l^{IA} is defined by $\delta_{IJ} l^{IA} l^{JB} := \gamma^{AB}$. Lowering the spacetime indices with $\tilde{g}_{ab}$ yields

$$\begin{cases} \tilde{e}^{+}_{a} = du_a \\ \tilde{e}^{-}_{a} = d\Omega_a - \alpha\Omega^2 du_a - \beta_A \Omega dx^A_a \\ \tilde{e}^{I}_{a} = l^I_A dx^A_a \,. \end{cases} \tag{7.10}$$

This choice for the tetrad $\{\tilde{e}^{\mu}_{a}\}$ defines the "orthonormal" (see (7.8)) basis at each point of the spacetime we will use. The advantage of using a tetrad system is that locally, at each point, we can work in the tetrad system as if we were in flat space.

7.1.3 Gamma Matrices

Using this tetrad system the gamma matrices σ^a in the curved spacetime can be easily defined as the Minkowski gamma matrices at each point in the tetrad system. It is not possible to define them in a coordinate independent ways but a choice of coordinates, in this case of the tetrad, has to be made. Let $\{\tilde{e}^{\mu}_{a}\}$ be a tetrad. At each point in spacetime the gamma matrices are defined in the tetrad corresponding to the point, i.e.,

$$\tilde{\sigma}^a := \sigma_\mu \tilde{e}^{\mu a} \,. \tag{7.11}$$

where σ_μ are the gamma matrices in flat spacetime. Using (7.9), an explicit expression is easily be found to be

$$\tilde{\sigma}^a = \sigma_+ \tilde{e}^{+a} + \sigma_- \tilde{e}^{-a} + \sigma_I \tilde{e}^{Ia} = \left(\sigma_+ + \Omega^2\alpha\sigma_- + \Omega\beta_A\tilde{\sigma}^A\right)\partial^a_\Omega + \sigma_-\partial^a_u + \tilde{\sigma}^A\partial^a_A \,, \tag{7.12}$$

where $\tilde{\sigma}^A := l^{IA}\sigma_I$. Lowering the index with (7.4) yields

$$\tilde{\sigma}_a = \sigma_- dr_a + \left(\sigma_+ - \alpha\Omega^2\sigma_-\right) du_a + \left(-\beta_A\Omega\sigma_- + \tilde{\sigma}_A\right) dx^A_a \,. \tag{7.13}$$

These are the expressions for the gamma matrices we will work with in the following to discuss the Witten equation which will be used, similarly to the Einstein equations above, to find some structure in the spinors we will be discussing. For this it is helpful to look at the different terms in (7.12), that is

$$\sigma_+ = \tilde{\sigma}^\Omega - \Omega^2\alpha\tilde{\sigma}^u - \Omega\beta_A\tilde{\sigma}^A \,, \tag{7.14}$$

$$\sigma_- = \tilde{\sigma}^u, \text{ and} \tag{7.15}$$

$$\tilde{\sigma}^A = l^{IA}\sigma_I \,. \tag{7.16}$$

In accordance with (7.1), we know that the gamma matrices in curved spacetime, σ_a, satisfy

$$\{\sigma_a, \sigma_b\} = 2g_{ab} I \,, \tag{7.17}$$

which can also easily be seen from the corresponding well-known relation for gamma matrices in flat spacetime

$$\{\sigma_\mu, \sigma_\nu\} = 2\lambda_{\mu\nu} I \tag{7.18}$$

using $\{\sigma_a, \sigma_b\} = \{\sigma_\mu, \sigma_\nu\} e^\mu_a e^\nu_b$ and (7.6). This can be used to show that the commutation relations

$$[\sigma_A, \sigma_\pm] = 2\sigma_A \sigma_\pm \,, \tag{7.19a}$$

and

$$[\sigma_A, \sigma_B] = -2\sigma_A \sigma_B + 2 s_{AB} I. \tag{7.19b}$$

hold. Furthermore, it is easy to check that

$$\sigma^a \sigma_a = d \,. \tag{7.20}$$

7.1.4 Projectors

We now define projectors on $\mathscr{S}(\mathscr{M})$ and a matrix representation for these and for the gamma matrices. The reason for introducing this will be explained in section 7.4. The elements

$$P_\pm := \frac{1}{2} \sigma_\pm \cdot \sigma_\mp \tag{7.21}$$

of $Cl(T\mathscr{M})$ are projectors since $P_+^2 = P_+$ and $P_-^2 = P_-$ while $P_+ P_- = P_- P_+ = 0$. The properties of $P_\pm$ follow directly from the definition (7.21) and the properties of the gamma matrices. The projectors decompose $\mathscr{S}(\tilde{\mathscr{M}})$ into two invariant subspaces $\mathscr{S}_\pm$. Thus, we can apply the projectors to the spinor and define

$$\psi_\pm := P_\pm \psi \,. \tag{7.22}$$

For the subsequent calculations it is advantageous to choose a representation of the gamma matrices and the projectors. As can be checked, a choice consistent with the above definitions is

$$\sigma_+ \doteq \sqrt{2} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- \doteq \sqrt{2} \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \qquad \text{and} \qquad \sigma_A \doteq \begin{pmatrix} \Gamma_A & 0 \\ 0 & -\Gamma_A \end{pmatrix} \tag{7.23}$$

for the gamma matrices, where $\{\Gamma_A, \Gamma_B\} = 2 s_{AB} I_{\mathscr{S}_\pm}$, and thus

$$P_+ \doteq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_- \doteq \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \tag{7.24}$$

for the projectors. We have

$$\Gamma^A \Gamma_A = d - 2 \,. \tag{7.25}$$

For the spinor we can write $\psi = (\psi_+, \psi_-)$. We conclude this subsection by collecting some relations we will need later, which can easily be verified using the representation chosen:

$$P_-\sigma_-\psi = \sigma_-P_+\psi = \sqrt{2}\begin{pmatrix}0\\ \psi_+\end{pmatrix} \tag{7.26a}$$

$$P_+\sigma_A\psi = \begin{pmatrix}0\\ \Gamma_A\psi_+\end{pmatrix} \tag{7.26b}$$

$$P_+\sigma_+\psi = \sigma_+P_-\psi = \sqrt{2}\begin{pmatrix}\psi_-\\ 0\end{pmatrix} \tag{7.26c}$$

$$P_-\sigma_+ = 0 = P_+\sigma_- \tag{7.26d}$$

Note that applying P_- to ψ yields ψ_- while applying it to $\sigma_-\psi$ gives ψ_+.

7.1.5 Spinor Connection

We adapt the spinor connection defined in Def. 3.30 to the present tetrad, where we follow [111], and derive the conformal transformation formula. For a vector field v^μ the covariant derivative is

$$\nabla_a v^b = \partial_a v^b + \Gamma^b_{ac} v^c\,. \tag{7.27}$$

Writing the vector in some tetrad, $v^\mu(x) = e^\mu_a(x)v^a(x)$, and applying the covariant derivative yields

$$\nabla_a v^\mu = \partial_a v^\mu + \omega^\mu_{a\nu} v^\nu \tag{7.28}$$

where $\omega_{a\mu\nu} := \frac{1}{8} e^b_\mu \nabla_a e_{\nu b}$ is a new connection similar to the Christoffel symbols in the Levi-Civita derivative. These two expressions for the covariant derivative have to be equal, i.e.,

$$\nabla_a v^\mu = e^{\mu b}\nabla_a v_b\,, \tag{7.29}$$

which will be the case if the spin connection is defined such that the covariant derivative of the tetrad is zero,

$$\nabla_a e^\mu_b = \partial_a e^\mu_b - \Gamma^c_{ab} e^\mu_c + \omega^\mu_{a\nu} e^\nu_b = 0\,. \tag{7.30}$$

In this case, ∇_a and $e^{\mu b}$ commute and we obviously have (7.29). There is enough information in (7.30) to uniquely determine the Christoffel symbols (leading to the usual formula) and the **spin connection** $\omega^{\mu\nu}_a$. Note that we still have $\nabla_a g_{bc} = 0$. The spinor field $\psi(x)$ is in the spinor representation of the Lorentz group so let $\mathcal{S}_{\mu\nu} := [\sigma_\mu, \sigma_\nu]$ be the generator of the Lorentz group in the spinor representation. Let ψ be a spinor and $\omega_a \equiv \omega^{\mu\nu}_a \mathcal{S}_{\mu\nu}$. The covariant derivative of the spinor is

$$\nabla_a\psi := \partial_a\psi + \omega_a\psi\,. \tag{7.31}$$

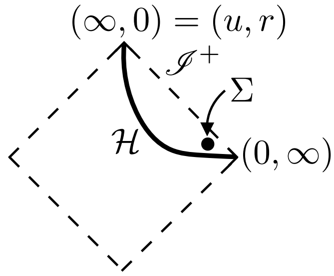


Figure 7.1: Sketch of the hypersurface $\mathcal{H}$ defined in (7.35). The $(d-2)$-dimensional surface Σ of constant (u, r) is represented by a point in this sketch. Here, $\mathcal{H}$ is sketched on the whole spacetime but we will consider it only close to infinity.

With this definition we find that under a change of the tetrad by $\Lambda(x)$ the covariant derivative transforms as $\nabla_a\psi(x) \to \Lambda(x)\nabla_a\psi(x)$ if we require that $\psi(x) \to \Lambda(x)\psi(x)$. Therefore, the physics does not change under Lorentz transformations as desired. The commutator of spin derivatives is [81]

$$[\nabla_a, \nabla_b]\psi = -\frac{1}{2}R_{abcd}\mathcal{S}^{cd}\psi. \tag{7.32}$$

We now turn to the conformal transformation of the spin derivative. The spin structure on $(\tilde{\mathscr{M}}, \tilde{g})$ is defined analogously to the one on $(\mathscr{M}, g)$ and on $\tilde{\mathscr{M}}$ the unphysical spinor $\tilde{\psi}$ is a section in $\tilde{\mathscr{S}}$. The physical and unphysical quantities are related by

$$\tilde{\psi} = \Omega^{1/2}\psi \quad \text{and} \quad \tilde{\sigma}_a = \Omega\sigma_a\,. \tag{7.33}$$

Therefore, as shown in appendix B, the physical and unphysical derivative are related by

$$\tilde{\nabla}_a\tilde{\psi} = r^{-1/2}\nabla_a\psi - \frac{1}{2r}\tilde{\sigma}_a\tilde{\sigma}^r\tilde{\psi}\,. \tag{7.34}$$

7.1.6 Witten Equation

Let ψ be a spinor. Consider the hypersurface defined by

$$\mathcal{H} = \left\{u = \frac{1}{2r}\right\} \tag{7.35}$$

near future null infinity, see Fig. 7.1. We assume that ψ fulfils the pair of equations

$$0 = \sum_{i=1}^{d-1} e_i \cdot \nabla_{e_i}\psi \tag{7.36a}$$

$$0 = \nabla_{e_0}\psi. \tag{7.36b}$$

on $\mathcal{H}$. Here, e_0 is the unit future timelike normal at $\mathcal{H}$ and $\{e_1, .., e_{d-1}\}$ is a positively oriented orthonormal basis of $T\mathcal{H}$ at each point. (7.36a) is called **Witten equation**. Note the similarity to the Dirac equation (3.31). Essentially, we split (3.31) up into a spacelike part, (7.36a), and a timelike part, (7.36b). The latter describes how ψ is extended off $\mathcal{H}$ (due to e_0 being the normal on $\mathcal{H}$) while the Witten equation restricts the spinor on $\mathcal{H}$. A spinor ψ which is a solution to the equations (7.36) will be called **Witten spinor**.

7.2 Outline of Proof

We have now all tools at hand we need to proof the positivity. More precisely, we will now proof

Theorem 7.1. *Assume that there is a Witten spinor on the hypersurface $\mathcal{H} = \left\{u = \frac{1}{2r}\right\}$ near infinity, that there is a Killing spinor on (Σ, s) and that the results from Lemma 6.1 hold. Then, $m_\Sigma \geq 0$.*

The calculations to establish the proof are rather lengthy and will take up the remainder of this chapter. However, the idea of the proof is relatively straightforward and similar the the one in four dimensions, see section 4.3. We start by showing that an integral at infinity over Q, defined in (7.3), is positive. The argument establishing this result is essentially the same as in four dimensions and we will only sketch the proof. It then remains to show that the integral over Q is equal to the Bondi mass. Finding this relation turns out to be significantly more difficult in higher dimensions than in four dimensions and the necessary calculations are the main part of the proof. The reason for this is again that the relevant terms of in the spinor expansion not of leading order (as in four dimensions) but "hidden" inside the asymptotic expansion. The remaining steps of the proof are as follows. First, the relation of Q and ψ at a given order of r is investigated by assuming an asymptotic expansion of both. The result is that, to relate Q to m_Σ, we need to find out more about the coefficients in the asymptotic expansion of ψ. This is a situation similar to the one in section 6.1 where we investigated the coefficients in the expansion of the metric coefficients. We also proceed similarly. Instead of the Einstein equations we now use the Witten equation to derive recursion relations. The results are summarized in Lemma 7.3 in section 7.4. This can be seen as an analogue of Lemma 6.1 in section 6.1. We proof Lemma 7.3 in the sections 7.5-7.8. The last step of the proof is the to use the results in Lemma 6.1 to show that the integral over Q is asymptotically equal to the Bondi mass. This is done in section 7.9 and shows that the Bondi mass is non-negative.

7.3 Positivity of Integral over Q

From the definition of Q we find

$$Q_{\mu\nu} = 2\Re\left\{\bar{\psi}\sigma_{[\nu}\nabla_{\mu]}\psi\right\} . \tag{7.37}$$

In the end, we want to show that an integral of Q over the boundary of $\mathcal{H}$, written $\partial\mathcal{H}$, is positive. However, $\mathcal{H}$ is non-compact which complicates the discussion, so we first look at the compact subset $C \subset \mathcal{H}$ and afterwards consider how the extension to all of $\mathcal{H}$ works. Q is a 2-form and $\dim \mathcal{H} = \dim C = d-1$ and thus $\dim \partial C = d-2$, so we can integrate

$$\int_C d \star Q = \int_{\partial C} \star Q \,, \tag{7.38}$$

where Stokes theorem was used. Thus, we want to show

Theorem 7.2. (7.38) *is positive if the dominant energy condition holds and if the spinor ψ used in the definition of Q fulfils the Witten equation.*

Sketch of Proof The proof of this statement is due to [14, 81] and we only sketch the idea here. For this, note that

$$Q_{\mu\nu} = 2\Re\left\{\bar{\psi}\sigma_{[\nu}\nabla_{\mu]}\psi\right\} = \nabla_{[\nu}w_{\mu]} \tag{7.39}$$

holds, where

$$w^\lambda := \bar{\psi}\sigma^\lambda\psi \tag{7.40}$$

is a non-spacelike vector as can be seen by computing $w^\nu w_\nu$. We take ψ to be a spinor fulfilling the Witten equation which means in particular that w^ν is divergence free, $\nabla_\nu w^\nu = 0$. Thus, we find for the divergence of Q the equation

$$\nabla^\mu Q_{\mu\nu} = \nabla^\mu\nabla_\mu w_\nu + 2\left[\nabla_\nu, \nabla_\mu\right] w^\mu \tag{7.41}$$

Using the commutator relation (7.32), the second term can be seen to give a contribution $\simeq G_{\nu\lambda}w^\lambda$ in the integral (7.38), where $G_{\mu\nu}$ is the Einstein tensor. Since w^μ is non-spacelike, this term provides a non-negative contribution if the Einstein equation holds and the dominant energy condition is fulfilled. Using some basic spinor identities it can be shown that the first term yields a positive contribution in the integral as well, one finds

$$0 \le \int_C d \star Q \,. \tag{7.42}$$

See, e.g., [81, 83, 112] for details of the arguments we sketched. □

It remains to be shown that the non-negativity holds not only for the compact hypersurface C but also for the hypersurface $\mathcal{H}$ we are actually interested in. For this, we follow the argument of [19]. The problem with replacing C by the non-compact $\mathcal{H}$ in (7.38) is that the integral might become divergent. To see that this is not the case we choose a r_0 such that $r \leq r_0 < \infty$ and consider a subset $\mathcal{H}(r_0) \subseteq \mathcal{H}$ defined by $\partial\mathcal{H}(r_0) = \Sigma(r_0) \equiv \Sigma(\frac{1}{2}r_0, r_0)$, where $\Sigma(u,r)$ is defined as in chapter 5. Hence, we know

$$0 \leq \int_{\Sigma(r_0)} \star Q \tag{7.43}$$

and we only need to show that taking the limit $r \to \infty$ does not lead to a divergent integral, that is we want to show

$$\lim_{r\to\infty} r^{d-2} \int_{\Sigma(r)} Q\left(\tilde{e}^+, \tilde{e}^-\right) dS_{\tilde{g}} \tag{7.44}$$

exists. Here, we evaluated $\star Q$ and chose the tetrad (7.9). $dS_{\tilde{g}}$ is the induced integration element on Σ. If there are future apparent horizons with boundaries $\mathscr{H}_i$ in the spacetime they are also part of the boundary $\partial\mathcal{H}(r_0) = \Sigma(r_0) \cup (\cup_i \mathscr{H}_i)$. Imposing on each $\mathscr{H}_i$ the boundary condition

$$(e_1 \wedge e_0) \cdot \psi = 2\psi\,, \tag{7.45}$$

where $e_1 \in T\mathcal{H}$ is the normal of $\mathscr{H}_i$ pointing outward, the contribution of each future apparent horizon to the integral over Q vanishes and we do not have to take it into account in the following [19, 87]. Therefore, we are at a point similar to theorem 6.2, where we had an integral which, upon taking the limit $r \to \infty$, is potentially divergent. The solution to the problem will also be similar. We will assume that there is an asymptotic expansion

$$Q \sim \sum_{n\in\mathbb{N}} Q^{(n)} r^{-n/2} \tag{7.46}$$

and then show that all terms of order $< d-2$ are total derivatives which vanish under the integral while terms of order $> d-2$ are falling off fast enough and will not contribute in the limit anyway. The crucial point is that the term $Q^{(2d-4)}$ is the only that does not vanish and, in fact, yields the Bondi mass which will then proof that the Bondi mass is non-negative. Hence, to conclude the proof it remains to be shown that the terms $Q^{(n)}$ have indeed this form. For this, we first take a closer look at the spinor ψ and, in particular, the additional structure we can find due to it being a solution of the Witten equation. This is done in the next section and is, by far, the longest part of the proof. The result can be used to find an explicit expression for the coefficients $Q^{(n)}$ thereby showing that the structure just described indeed exists and that the integral (7.46) converges with the limit being the Bondi mass.

7.4 Spinor Recursion Relation

In this section we assume that the spinor ψ, which is a solution to the Witten equation, has an asymptotic expansion. Then, recursively, we show that a lot of coefficients in the series vanish, similar to the results in Lemma 6.1. Recall that there is the relation $\tilde{\psi} = r^{1/2}\psi$ between the physical and unphysical spinor. The ansatz we make for the asymptotic expansion on $\mathcal{H}$ is

$$\psi \sim r^{-\frac{1}{2}} \sum_{n\in\mathbb{N}} \psi^{(n)}(r,u,x^A)\, r^{-n/2}\,, \tag{7.47}$$

where we assume that each $\psi^{(n)}$ is smooth and satisfies

$$\tilde{\nabla}_\Omega \psi^{(n)} = 0 \tag{7.48}$$

near null infinity. We will discuss this assumptions/condition at the end of section 7.5. The lemma we want to proof is

Lemma 7.3. *Let ψ be a smooth Witten spinor with asymptotic expansion* (7.47)*. Assume that $\psi_+^{(0)} = \epsilon$ is a Killing spinor on Σ and that $\psi_-^{(0)} = 0$. Then, for $1 \le n \le 2d-3$ and $n \neq d$:*

$$\begin{aligned}\psi_-^{(n)} = \frac{\sqrt{2}}{d-n}\Bigg\{&\Gamma^A\mathcal{D}_A\psi_+^{(n-2)} + \frac{1}{2}\gamma_{AB}^{(n-2)}\Gamma^B\mathcal{D}^A\psi_+^{(0)} + \frac{n-4d}{16}\beta_A^{(n-2)}\Gamma^A\psi_+^{(0)}\\ &+\frac{1}{4}\Gamma^A\left[\mathcal{D}^B\gamma_{AB}^{(n-2)} - \mathcal{D}_A\gamma^{(n-2)}\right]\psi_+^{(0)} + \frac{n-2}{\sqrt{2}}\gamma^{(n-2)}\psi_-^{(2)}\Bigg\}\end{aligned} \tag{7.49}$$

$$\begin{aligned}\psi_+^{(n)} = \frac{-1}{\sqrt{2}n}\Bigg\{&\frac{10-n-2d}{16}\beta_A^{(n-4)}\Gamma^A\psi_-^{(2)} - \frac{6-d-n}{\sqrt{2}}\alpha^{(0)}\psi_+^{(n-4)} + \beta_A^{(n-4)}\Gamma^A\psi_-^{(2)} - \beta^{A(n-4)}\sqrt{2}\mathcal{D}_A\psi_+^{(0)}\\ &+\Gamma^A\mathcal{D}_A\psi_-^{(n-2)} - \frac{1}{\sqrt{2}}\beta^{A(n-4)}\mathcal{D}_A\psi_+^{(0)} + \frac{1}{2}\Gamma^C\gamma_{BC}^{(n-4)}\mathcal{D}^B\psi_-^{(2)} + \frac{1}{4}\left[\mathcal{D}^B\gamma_{AB}^{(n-4)} - \mathcal{D}_A\gamma^{(n-4)}\right]\Gamma^A\psi_-^{(2)}\\ &+\frac{1}{2\sqrt{2}}\gamma'^{(n-2)}\psi_+^{(0)} + \frac{1}{\sqrt{2}}\alpha^{(0)}\frac{n-4}{n-2}\gamma^{(n-4)}\psi_+^{(0)} + \frac{1}{2\sqrt{2}}\mathcal{D}_A\beta_B^{(n-4)}\Gamma^A\Gamma^B\psi_+^{(0)}\\ &+\left(1-\frac{d}{2}\right)\sqrt{2}\alpha^{(n-4)}\psi_+^{(0)}\Bigg\}\,.\end{aligned} \tag{7.50}$$

For $n = d$ only the second equation holds while the first equation does not hold and the expression in the bracket has to vanish. For $n = 2d-2$, both equations still hold but there are additional terms; the terms

$$+\frac{n-2}{4}\gamma^{AB(n/2-1)}\beta_B^{(n/2-1)}\sqrt{2}\mathcal{D}_A\psi_+^{(0)} - \frac{1}{4}s^{DC}\gamma_{AC}'^{(n/2-1)}\gamma_{BD}^{(n/2-1)}\Gamma^A\Gamma^B\sqrt{2}\psi_+^{(0)} + \frac{d-2}{4}\Gamma^B\beta^{A(d-2)}\gamma_{AB}^{(d-2)}\psi_-^{(0)}\,, \tag{7.51}$$

have to be added in the bracket on the right-hand side of (7.49) *and the terms*

$$+\frac{1}{4}\gamma_{BC}^{(d-2)}\gamma^{AC(d-2)}\Gamma^B\mathcal{D}_A\psi_+^{(0)} + \frac{30d-23d^2}{64(d-2)}\beta^{C(d-2)}\gamma_{BC}^{(d-2)}\Gamma^B\psi_+^{(0)} + \frac{n-2}{16\sqrt{2}}\gamma_{AB}^{(n/2-1)}\gamma^{(n/2-1)AB}\psi_-^{(2)} \tag{7.52}$$

have to be added in the bracket on the right-hand side of (7.50).

The proof of the lemma is rather lengthy and we split it up into several sections. First, in section 7.5, we find an explicit expression of the Witten equation using the tetrad (7.9). Second, some auxiliary calculations are collected in section 7.6 for future reference. Third, we write down the Witten equation at a given order $r^{-n/2}$ in section 7.7 This is then, in the fourth and last step in section 7.8, used to derive the equations stated in Lemma 7.3.

7.5 Proof Lemma 7.3— Step 1: Witten Equation

To find an explicit form of (7.36a), it is advantageous to first look at (7.36b) and use this equation to find an expression for $\tilde{\nabla}_u$ in terms of $\tilde{\nabla}_r$ and $\tilde{\nabla}_A$. In this and the following section indices are raised/lowered by γ_{AB}.

7.5.1 Equation (7.36b)

Define

$$f = -u + \frac{1}{2r} \tag{7.53}$$

such that the hypersurface (7.35) is defined by the condition $f = 0$. The normal n_a to this surface can be found with the standard formula $n_a := \partial_a f$. Raising the index we find

$$\begin{aligned} n^\Omega &= g^{\Omega\Omega} n_\Omega + g^{\Omega u} n_u = \frac{\Omega^4}{2}\left(2\alpha + \beta^A\beta_A\right) - \Omega^2 \\ n^u &= g^{u\Omega} n_\Omega = \frac{\Omega^2}{2} \\ n^A &= g^{A\Omega} n_\Omega = \frac{\Omega^3}{2}\beta^A \end{aligned} \tag{7.54}$$

Now take the e_0 appearing in (7.36b) to be this n^a. Then after performing the conformal transformation of ∇_a (c.f. (7.34)) the equation (7.36b) reads

$$\nabla_n = n^a\nabla_a = n^a\tilde{\nabla}_a - \frac{r}{2} n^a\tilde{\sigma}_a\tilde{\sigma}^b\tilde{\nabla}_b r^{-1} = n^a\tilde{\nabla}_a - \frac{r}{2} n^a\tilde{\sigma}_a\tilde{\sigma}^\Omega \tag{7.55}$$

where ψ was dropped for the moment. This equation of differential operators holds once it acts on ψ from the left. Using the expressions for the gamma matrices in the chosen tetrad we found in (7.12) and (7.13) we have the relations

$$n^\Omega\tilde{\sigma}_\Omega\tilde{\sigma}^\Omega = \left[\frac{\Omega^4}{2}\left(2\alpha + \beta^A\beta_A\right) - \Omega^2\right]\left[\sigma_-\sigma_+ + \frac{1}{r}\beta_A\sigma_-\tilde{\sigma}^A\right], \tag{7.56a}$$

$$n^u\tilde{\sigma}_u\tilde{\sigma}^\Omega = \frac{\Omega^2}{2}\left[\Omega^2\alpha\sigma_+\sigma_- + \Omega\beta_A\sigma_+\tilde{\sigma}^A - \alpha\Omega^2\sigma_-\sigma_+ - \alpha\Omega^3\beta_A\sigma_-\tilde{\sigma}^A\right], \tag{7.56b}$$

and

$$n^A\tilde{\sigma}_A\tilde{\sigma}^\Omega = \frac{\Omega^3}{2}\beta^A\left[-\Omega\beta_A\sigma_-\sigma_+ - \Omega^2\beta_A\beta_B\sigma_-\tilde{\sigma}^B + \tilde{\sigma}_A\sigma_+ + \Omega^2\alpha\tilde{\sigma}_A\sigma_- + \Omega\beta_B\tilde{\sigma}_A\tilde{\sigma}^B\right]. \tag{7.56c}$$

Adding the equations (7.56) together yields

$$\begin{aligned} n^a\tilde{\sigma}_a\tilde{\sigma}^\Omega = {} & \sigma_-\sigma_+\left[-\Omega^2 + \frac{1}{2}\alpha\Omega^4\right] + \frac{\Omega^4}{2}\alpha\sigma_+\sigma_- + \sigma_-\tilde{\sigma}^A\left[\frac{\Omega^5}{2}\alpha\beta_A - \Omega^3\beta_A\right] \\ & + \frac{\Omega^5}{2}\alpha\beta^A\tilde{\sigma}_A\sigma_- + \frac{\Omega^3}{2}\beta^A\sigma_+\tilde{\sigma}^A + \frac{\Omega^3}{2}\beta^A\tilde{\sigma}^A\sigma_+ + \frac{\Omega^4}{2}\beta^A\beta_B\tilde{\sigma}_A\tilde{\sigma}^B. \end{aligned} \tag{7.57}$$

Using the (anti)commutation relations, see (7.18) and (7.19), this simplifies to

$$n^a\tilde{\sigma}_a\tilde{\sigma}^\Omega = \Omega^4\alpha - \Omega^2\sigma_-\sigma_+ - \Omega^3\beta_A\sigma_-\tilde{\sigma}^A + \frac{\Omega^4}{2}\beta^A\beta_A. \tag{7.58}$$

This can be plugged into (7.55) and, after some small manipulations, we find

$$\sigma_-\tilde{\nabla}_u\tilde{\psi} = (2 - 2\Omega^2\alpha - \Omega^2\beta_A\beta^A)\tilde{\nabla}_\Omega\tilde{\psi} - \Omega\beta^A\sigma_-\tilde{\nabla}_A\tilde{\psi} + \left(\Omega^2\alpha + \frac{1}{2}\Omega^2\beta_A\beta^A\right)\sigma_-\tilde{\psi}. \tag{7.59}$$

Hence, we have an expression which will be used to replace all $\tilde{\nabla}_u$ occurring in the Witten equation we will turn to now.

7.5.2 Rewriting the Witten Equation

An explicit expression of the Witten equation (7.36a) is derived now by using the tetrad system and corresponding gamma matrices introduced above. Using the relation $\tilde{\sigma}_a = \Omega\sigma_a$ relating physical and unphysical gamma matrices and the conformal transformation of $\nabla_b\psi$, see (7.34), we have

$$0 = \sigma^b\nabla_b\psi = r^{-1}\tilde{\sigma}^b\nabla_b\psi = r^{-1/2}\tilde{\sigma}^b\left[\tilde{\nabla}_b\tilde{\psi} - \frac{1}{2}r\tilde{\sigma}_b\tilde{\sigma}_a\left(\tilde{\nabla}^a r^{-1}\right)\tilde{\psi}\right] \tag{7.60}$$

and therefore the Witten equation takes the form

$$0 = \tilde{\sigma}^b\tilde{\nabla}_b\tilde{\psi} + \frac{d}{2r}\tilde{\sigma}^r\tilde{\psi}. \tag{7.61}$$

Substituting (7.12) yields

$$0 = \left(\sigma_+ + \Omega^2\alpha\sigma_- + \Omega\beta_A\tilde{\sigma}^A\right)\tilde{\nabla}_\Omega\tilde{\psi} + \sigma_-\tilde{\nabla}_u\tilde{\psi} + \tilde{\sigma}^A\tilde{\nabla}_A\psi + \frac{d}{2r}\tilde{\sigma}^\Omega\tilde{\psi} \tag{7.62}$$

and now we can use (7.59) to replace $\tilde{\nabla}_u$, i.e.,

$$\begin{aligned} 0 = {} & \left(\sigma_+ + \Omega^2\alpha\sigma_- + \Omega\beta_A\tilde{\sigma}^A\right)\tilde{\nabla}_\Omega\tilde{\psi} + \left[2 - 2\Omega^2\alpha - \Omega^2\beta^A\beta_A\right]\sigma_-\tilde{\nabla}_\Omega\tilde{\psi} - \Omega\beta^A\sigma_-\tilde{\nabla}_A\tilde{\psi} \\ & + \Omega^2\alpha\sigma_-\tilde{\psi} + \tilde{\sigma}^A\tilde{\nabla}_A\psi + \frac{d}{2r}\left(\sigma_+ + \Omega^2\alpha\sigma_- + \Omega\beta_A\tilde{\sigma}^A\right)\tilde{\psi}. \end{aligned} \tag{7.63}$$

Hence, an explicit form of the Witten equation is

$$
\begin{aligned}
0 = {} & \Omega\sigma_+\tilde{\nabla}_\Omega\tilde{\psi} + \left(2 - \Omega^2\alpha - \Omega^2\beta_A\beta^A\right)\sigma_-\Omega\tilde{\nabla}_\Omega\tilde{\psi} - \Omega^2\beta_A\tilde{\sigma}^A\tilde{\nabla}_\Omega\tilde{\psi} - \Omega^2\beta^A\sigma_-\tilde{\nabla}_A\tilde{\psi} + \Omega\tilde{\sigma}^A\tilde{\nabla}_A\tilde{\psi} \\
& - \frac{d}{2}\sigma_+\tilde{\psi} + \frac{(2-d)\Omega^2\alpha}{2}\sigma_-\tilde{\psi} - \frac{d}{2}\Omega\beta_A\tilde{\sigma}^A\tilde{\psi}\,.
\end{aligned} \tag{7.64}
$$

This is the equation we will work with for the remainder of the proof. Our goal is to treat this equation similarly to the Einstein equations in the previous chapter. That is, we substitute the asymptotic expansion of ψ, i.e. (7.47), into (7.64) and try to solve the equation recursively at each order of r. However, we see that a problem/difference to the case of the Einstein equations occurs. Namely, the coefficients $\psi^{(k)}(r, u, x^A)$ in the asymptotic expansion of the spinor still depend on r and are not independent of it as was the case for the asymptotic expansion of the metric coefficients in (5.14). In particular, the series is not unique. We can only assume that $\tilde{\nabla}_\Omega\psi^{(k)} = 0$ which contains information about the parallel transport of the coefficients $\psi^{(k)}$ in r-direction near null infinity. This has the following consequence. We cannot simply plug in the asymptotic expansion into the Witten equation and read off the equation corresponding to a given order of r since there will always be factors of r hidden inside $\psi^{(k)}$. What will be done instead is applying the covariant derivative n-times to (7.64). In particular, the derivatives will act on $\psi^{(k)}$ and then we can use (7.48) such that these terms vanish. This ensures that powers of r hidden inside $\psi^{(k)}$ are taken into account appropriately. Unfortunately, the gamma matrices also depend on r (since the tetrad does) and thus the derivatives also acts non-trivially on them and, additionally, $\tilde{\nabla}_\Omega$ and $\tilde{\nabla}_A$ do not commute. This means that after applying the covariant derivative to (7.64), we cannot directly apply (7.48) but we first have to consider the action of the derivative on the gamma matrices (which are always to the left of the spinor in (7.64)) and the commutator of $\tilde{\nabla}_\Omega$ and $\tilde{\nabla}_A$. This complicates the computation considerably since the expressions arising are rather lengthy. Thus, we will do all auxiliary calculations in the following section and collect all expressions (derivative of gamma matrices, commutator, ...) that are needed later there. Afterwards we return to the Witten equation and, using the results of the next section, apply the covariant derivative.

7.6 Proof Lemma 7.3— Step 2: Auxiliary Calculations

To find the Witten equation at a give order we need the nth derivative of the gamma matrices as well as the commutator of $\tilde{\nabla}^n_\Omega$ and $\tilde{\nabla}_A$, we begin with the latter.

7.6.1 Commutator $[\tilde{\nabla}^n_\Omega, \tilde{\nabla}_A]$

The general formula for the commutator of the derivative is

$$\phi_{\mu\nu} := \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu = \frac{1}{8} R_{\mu\nu\alpha\beta}[\sigma^\alpha, \sigma^\beta] \,. \tag{7.65}$$

This is the only commutator we need to compute since, by induction, one can easily show that the commutator of the nth derivative with respect to Ω is

$$\tilde{\nabla}^n_\Omega \tilde{\nabla}_A \psi = \tilde{\nabla}_A \tilde{\nabla}^n_\Omega \psi + \sum_{k=1}^{n} \binom{n}{k} \left(\tilde{\nabla}^{n-k}_\Omega \psi\right) \left(\tilde{\nabla}^{k-1}_\Omega \phi_{\Omega A}\right) \,. \tag{7.66}$$

The proof is simply by induction. Thus, we want to compute

$$\begin{aligned} 8\phi_{\Omega A} &= \tilde{g}_{\Omega\lambda} \tilde{R}^\lambda{}_{A\alpha\beta}[\tilde{\sigma}^\alpha, \tilde{\sigma}^\beta] \\ &= 2g_{\Omega\lambda}\left(\tilde{R}^\lambda{}_{A\Omega u}[\tilde{\sigma}^\Omega, \tilde{\sigma}^u] + \tilde{R}^\lambda{}_{A\Omega B}[\tilde{\sigma}^\Omega, \tilde{\sigma}^B] + \tilde{R}^\lambda{}_{AuB}[\tilde{\sigma}^u, \tilde{\sigma}^B]\right) + g_{\Omega\lambda}\tilde{R}^\lambda{}_{ABC}[\tilde{\sigma}^B, \tilde{\sigma}^C] \,. \end{aligned} \tag{7.67}$$

The commutators of the gamma matrices we need are

$$\begin{aligned} &[\tilde{\sigma}^\Omega, \tilde{\sigma}^u] = [\sigma_+ + \Omega^2\alpha\sigma_- + \Omega\beta_A\tilde{\sigma}^A, \sigma_-] = [\sigma_+, \sigma_-] + \Omega\beta^A[\tilde{\sigma}^A, \sigma_-] \,, \\ &[\tilde{\sigma}^\Omega, \tilde{\sigma}^B] = [\sigma_+, \tilde{\sigma}^B] + \Omega^2\alpha[\sigma_-, \tilde{\sigma}^B] + \Omega\beta_A[\tilde{\sigma}^A, \tilde{\sigma}^B] \,, \text{ and} \\ &[\tilde{\sigma}^u, \tilde{\sigma}^B] = [\sigma_-, \tilde{\sigma}^B] \,. \end{aligned} \tag{7.68}$$

Additionally, we have to compute

$$\begin{aligned} \tilde{g}_{\Omega\lambda}\tilde{R}^\lambda{}_{A\Omega u} &= \tilde{g}_{\Omega u}\tilde{R}^u{}_{A\Omega u} = \tilde{R}^u{}_{A\Omega u} \,, \\ \tilde{g}_{\Omega\lambda}\tilde{R}^\lambda{}_{A\Omega B} &= \tilde{R}^u{}_{A\Omega B} \,, \\ \tilde{g}_{\Omega\lambda}\tilde{R}^\lambda{}_{AuB} &= \tilde{R}^u{}_{AuB} \,, \text{ and} \\ \tilde{g}_{\Omega\lambda}\tilde{R}^\lambda{}_{ABC} &= \tilde{R}^u{}_{ABC} \,, \end{aligned}$$

so four components of the Riemann tensor are needed. From the Christoffel symbols one finds by explicit calculation

$$\tilde{R}^u_{A\Omega u} = -\frac{1}{4}\partial_\Omega(\Omega\beta_B)\gamma^{BC}\partial_\Omega\gamma_{AC} + \frac{1}{2}\partial^2_\Omega(\Omega\beta_A) \,, \tag{7.69a}$$

$$\tilde{R}^u_{ArB} = -\frac{1}{2}\partial^2_\Omega(\gamma_{AB}) + \frac{1}{4}\gamma^{CD}\partial_\Omega(\gamma_{AC})\partial_\Omega(\gamma_{BD}) \,, \tag{7.69b}$$

$$\begin{aligned} \tilde{R}^u_{AuB} = &-\frac{1}{2}\partial_u\partial_\Omega(\gamma_{AB}) - \frac{1}{2}\partial_A\partial_\Omega(\Omega\beta_B) - \frac{1}{2}\partial_\Omega(\Omega^2\alpha)\partial_\Omega\gamma_{AB} + \frac{1}{4}\partial_\Omega(\Omega\beta_C)\cdot \\ &\left(2\Gamma^C_{AB} - \Omega\beta^C\partial_\Omega\gamma_{AB}\right) - \frac{1}{4}\partial_\Omega(\Omega\beta_A)\partial_\Omega(\Omega\beta_B) + \frac{1}{4}\partial_\Omega(\gamma_{BD})\cdot \\ &\left(\gamma^{DC}\partial_u\gamma_{AC} - \Omega\beta^D\partial_\Omega(\Omega\beta_A)\right) \,, \end{aligned} \tag{7.69c}$$

and

$$\tilde{R}^u_{ABC} = \partial_{[B}\partial_{|\Omega|}\gamma_{C]A} + \frac{1}{2}\partial_\Omega(\Omega\beta_{[C})\partial_{|\Omega|}\gamma_{B]A} + \partial_\Omega(\gamma_{D[B})\Gamma^D_{C]A}\,, \tag{7.69d}$$

where $\Gamma^A_{BC} = \frac{1}{2}\gamma^{AD}(\partial_B\gamma_{CD} + \partial_C\gamma_{BD} - \partial_D\gamma_{BC})$ was used to keep the expressions shorter.

Now, we have all terms needed to calculate (7.67). Simply substituting the results for the coefficients of the Riemann tensor and the commutators of the gamma matrices yields

$$\begin{aligned}
8\phi_{\Omega A} &= \Big\{-\frac{1}{2}\partial_\Omega(\Omega\beta_B)\gamma^{BC}\partial_\Omega\gamma_{AC} + \partial^2_\Omega(\Omega\beta_A)\Big\}\cdot\Big\{[\sigma_+,\sigma_-] + \Omega\beta^A[\tilde{\sigma}^A,\sigma_-]\Big\}\\
&+\Big\{-\partial^2_\Omega(\gamma_{AB}) + \frac{1}{2}\gamma^{CD}\partial_\Omega(\gamma_{AC})\partial_\Omega(\gamma_{BD})\Big\}\cdot\Big\{[\sigma_+,\tilde{\sigma}^B] + \Omega^2\alpha[\sigma_-,\tilde{\sigma}^B] + \Omega\beta_A[\tilde{\sigma}^A,\tilde{\sigma}^B]\Big\}\\
&+\Big\{-\partial_u\partial_\Omega(\gamma_{AB}) - \partial_A\partial_\Omega(\Omega\beta_B) - \partial_\Omega(\Omega^2\alpha)\partial_\Omega\gamma_{AB} + \frac{1}{2}\partial_\Omega(\Omega\beta_C)\left(2\Gamma^C_{AB} - \Omega\beta^C\partial_\Omega\gamma_{AB}\right)\\
&-\frac{1}{2}\partial_\Omega(\Omega\beta_A)\partial_\Omega(\Omega\beta_B) + \frac{1}{2}\partial_\Omega(\gamma_{BD})\left(\gamma^{DC}\partial_u\gamma_{AC} - \Omega\beta^D\partial_\Omega(\Omega\beta_A)\right)\Big\}\cdot[\sigma_-,\tilde{\sigma}^B]\\
&+\Big\{\partial_{[B}\partial_{|\Omega|}\gamma_{C]A} + \frac{1}{2}\partial_\Omega(\Omega\beta_{[C})\partial_{|\Omega|}\gamma_{B]A} + \partial_\Omega(\gamma_{D[B})\Gamma^D_{C]A}\Big\}\cdot[\tilde{\sigma}^B,\tilde{\sigma}^C]\,.
\end{aligned}$$

7.6.2 Derivative of Gamma Matrices

Now, the final ingredient needed is the action of $\tilde{\nabla}^n_\Omega$ on the gamma matrices. The first derivatives with respect to Ω are as follows. For the r-component we find

$$\begin{aligned}
\tilde{\nabla}_\Omega\tilde{\sigma}^\Omega &= \tilde{\nabla}_\Omega\left(\tilde{\sigma}^b d\Omega_b\right) = \tilde{\sigma}^b\left(\partial_\Omega d\Omega_b - \tilde{\Gamma}^c_{\Omega b}d\Omega_c\right) = -\tilde{\sigma}^b\tilde{\Gamma}^\Omega_{\Omega b} = -\tilde{\sigma}^\Omega\tilde{\Gamma}^\Omega_{\Omega\Omega} - \tilde{\sigma}^u\tilde{\Gamma}^\Omega_{\Omega u} - \tilde{\sigma}^A\tilde{\Gamma}^\Omega_{\Omega A}\\
&= \frac{1}{2}\left[2\partial_\Omega(\Omega^2\alpha) + \Omega\beta^A\partial_\Omega(\Omega\beta_A)\right]\tilde{\sigma}^u + \frac{1}{2}\left[\beta_A + \Omega(\partial_\Omega\beta_A - \beta^B\partial_\Omega\gamma_{AB})\right]\tilde{\sigma}^A
\end{aligned}\tag{7.70a}$$

while the u-component vanishes since

$$\tilde{\nabla}_\Omega\tilde{\sigma}^u = -\tilde{\sigma}^b\tilde{\Gamma}^u_{\Omega b} = 0\,. \tag{7.70b}$$

The derivative of the A-component is again non-trivial,

$$\tilde{\nabla}_\Omega\tilde{\sigma}^A = -\tilde{\sigma}^b\tilde{\Gamma}^A_{\Omega b} = \frac{1}{2}\gamma^{AB}\partial_\Omega(\Omega\beta_B)\tilde{\sigma}^u - \frac{1}{2}\gamma^{AC}\partial_\Omega\gamma_{BC}\tilde{\sigma}^B\,. \tag{7.70c}$$

Using this we can find the derivative of σ_+ which is the component that actually appears in the Witten equation. We have

$$\begin{aligned}
\tilde{\nabla}_\Omega\sigma_+ &= \tilde{\nabla}_\Omega\left(\tilde{\sigma}^\Omega - \Omega^2\alpha\tilde{\sigma}^u - \Omega\beta_A\tilde{\sigma}^A\right) = \tilde{\nabla}_\Omega\tilde{\sigma}^\Omega - \tilde{\nabla}_\Omega(\Omega^2\alpha)\tilde{\sigma}^u - \tilde{\nabla}_\Omega(\Omega\beta_A\tilde{\sigma}^A)\\
&= -\frac{1}{2}\partial_\Omega\left(\Omega\beta_A\right)\tilde{\sigma}^A\,.
\end{aligned}\tag{7.70d}$$

The derivative of σ_- is trivial since $\sigma_- = \tilde{\sigma}^u$. This concludes this section collecting the results of auxiliary calculations we will need when investigating the Witten equation which we will do now.

7.7 Proof Lemma 7.3— Step 3: Witten Equation at Order $r^{-n/2}$

Now, we can go back to the Witten equation (7.64) which, neglecting the irrelevant term with $\beta_A\beta^A$, reads

$$0 = \Omega\sigma_+\tilde{\nabla}_\Omega\tilde{\psi} + \left(2\Omega - \Omega^3\alpha\right)\sigma_-\tilde{\nabla}_\Omega\tilde{\psi} + \Omega^2\beta_A\tilde{\sigma}^A\tilde{\nabla}_\Omega\tilde{\psi} - \Omega^2\beta^A\sigma_-\tilde{\nabla}_A\tilde{\psi} + \Omega\tilde{\sigma}^A\tilde{\nabla}_A\tilde{\psi}$$
$$- \frac{d}{2}\sigma_+\tilde{\psi} + \frac{(2-d)\Omega^2\alpha}{2}\sigma_-\tilde{\psi} - \frac{d}{2}\Omega\beta_A\tilde{\sigma}^A\tilde{\psi}\,. \tag{7.71}$$

We assumed that the physical spinor has asymptotic expansion (7.47) and thus the unphysical spinor $\tilde{\psi} = r^{1/2}\psi$ has asymptotic expansion

$$\tilde{\psi} \sim \sum_{n\in\mathbb{N}} \psi^{(n)} r^{-n/2}\,. \tag{7.72}$$

This can be plugged into the Witten equation. Then we look at the equation at each order of r by applying $\tilde{\nabla}_\Xi$, where $\Xi = \sqrt{\Omega}$, n times where we assume that $0 \le n \le 2d-2$. Additionally, we will now use the results in Lemma 6.1. Since the calculations are rather lengthy we will look at each term individually. The first term is

$$\mathrm{I} := \sum_{k=0}^{n} \binom{n}{k} \tilde{\nabla}_\Xi^k\left(\sigma_+\right)\tilde{\nabla}_\Xi^{n-k}\left(\Omega\tilde{\nabla}_\Omega\tilde{\psi}\right)\,.$$

We have

$$X := \Omega\tilde{\nabla}_\Omega\tilde{\psi} = \sum_{i\in\mathbb{N}} -\frac{i}{2}\Omega^{i/2}\psi^{(i)} + \Omega^{i/2}\tilde{\nabla}_r\psi^{(i)} \tag{7.73}$$

where we substituted the asymptotic expansion for $\tilde{\psi}$. Thus, using $\tilde{\nabla}_\Xi\sigma_+ = 2\Xi\tilde{\nabla}_\Omega\sigma_+$ and the relevant expression derived above, that is

$$\tilde{\nabla}_\Omega\sigma_+ = -\frac{1}{2}\partial_\Omega\left(\Omega\beta_A\right)\,,$$

we find

$$\begin{aligned}
\mathrm{I} &= \sigma_+\tilde{\nabla}_\Xi^n X + \sum_{k=1}^{n}\binom{n}{k}\tilde{\nabla}_\Xi^k\left(\sigma_+\right)\tilde{\nabla}_\Xi^{n-k}X \\
&= \sigma_+\tilde{\nabla}_\Xi^n X + \sum_{k=1}^{n}\binom{n}{k}\tilde{\nabla}_\Xi^{k-1}\left[-\Xi\partial_\Omega\left(\Omega\beta_A\right)\tilde{\sigma}^A\right]\tilde{\nabla}_\Xi^{n-k}X \\
&= \sigma_+\tilde{\nabla}_\Xi^n X - \sum_{k=1}^{n}\binom{n}{k}\tilde{\nabla}_\Xi^{k-1}\left[\Xi\partial_\Omega(\Omega\beta_A)\right]\tilde{\sigma}^A\tilde{\nabla}_\Xi^{n-k}X + O(\text{higher order terms})\,,
\end{aligned} \tag{7.74}$$

where "higher order terms" refers to terms including factors of $\beta^A \partial_\Omega(\Omega\beta_A)$ or $\beta^B \partial_\Omega \gamma_{AB}$. Now, we only need to substitute the asymptotic expansion (5.14) of the metric coefficients, take $r \to \infty$ (by which the higher order terms vanish) and use $\tilde{\nabla}_\Omega \psi^{(i)} = 0$. Doing this leads to

$$\mathrm{I}|_{r\to\infty} \simeq \frac{n}{2} n!\, \sigma_+ \psi^{(n)} - n! \tilde{\sigma}^A \sum_{k=1}^{n} \frac{n-k}{4} \beta_A^{(k-2)} \psi^{(n-k)} . \tag{7.75}$$

The next term is

$$\mathrm{II} := \sum_{k=0}^{n} \binom{n}{k} \partial_\Xi^k (2 - \Omega^2\alpha)\sigma_- \tilde{\nabla}_\Xi^{n-k}(\Omega\tilde{\nabla}_\Omega\tilde{\psi}) = 2\tilde{\nabla}_\Xi^n(\Omega\tilde{\nabla}_\Omega\tilde{\psi}) - \sum_{k=0}^{n} \binom{n}{k} \partial_\Xi^k(\Omega^2\alpha)\sigma_- \tilde{\nabla}_\Xi^{n-k}(\Omega\tilde{\nabla}_\Omega\tilde{\psi}) ,$$

where we again used that all derivatives of $\tilde{\sigma}_-$ vanish. Substituting the asymptotic expansions of the spinor and α, taking the limit $r \to \infty$ this is

$$\mathrm{II}|_{r\to\infty} \simeq n n! \sigma_- \psi^{(n)} - n! \sigma_- \sum_{k=1}^{n-4} \frac{k}{2} \alpha^{(n-4-k)} \psi^{(k)} . \tag{7.76}$$

Continuing with

$$\mathrm{III} := \sum_{k=0}^{n} \binom{n}{k} \tilde{\nabla}_\Xi^k \left(\beta_A \tilde{\sigma}^A\right) \tilde{\nabla}_\Xi^{n-k}(\Omega^2 \tilde{\nabla}_\Omega \tilde{\psi})$$

we need to use (7.70c) for the derivatives acting on $\tilde{\sigma}^A$ but all terms which enter this way will vanish in the end since their order is too high, as can easily been seen, so keeping only relevant terms yields

$$\mathrm{III} \simeq \sum_{k=0}^{n} \binom{n}{k} \partial_\Xi^k \left(\beta_A\right) \tilde{\sigma}^A \tilde{\nabla}_\Xi^{n-k}(\Omega^2 \tilde{\nabla}_\Omega \tilde{\psi}) .$$

The same steps as before lead to

$$\mathrm{III}|_{r\to\infty} \simeq n! \sum_{m=0}^{n-3} \frac{n-m-2}{2} \beta_A^{(m)} \tilde{\sigma}^A \psi^{(n-m-2)} . \tag{7.77}$$

The fourth term is

$$\mathrm{IV} := -\sum_{k=0}^{n} \binom{n}{k} \tilde{\nabla}_\Xi^k \left(\Omega^2 \beta^A \sigma_-\right) \tilde{\nabla}_\Xi^{n-k} \tilde{\nabla}_A \tilde{\psi} .$$

and we find

$$\mathrm{IV}|_{r\to\infty} \simeq -n! \sum_{m=0}^{n-4} \beta^{A(n-4-m)} \sigma_- \mathcal{D}_A \psi^{(m)} . \tag{7.78}$$

with an additional term $+\frac{n-2}{4} n! \gamma^{AB(n/2-1)} \beta_B^{(n/2-1)} \sigma_- \mathcal{D}_A \psi^{(0)}$ if $n = 2d-2$ The fifth term reads

$$\mathrm{V} := \tilde{\nabla}_\Xi^n \left(\Omega \tilde{\sigma}^A \tilde{\nabla}_A \tilde{\psi}\right) .$$

Since

$$\begin{aligned}
\mathrm{V} &= \sum_{k=0}^{n} \binom{n}{k} \tilde{\nabla}_{\Xi}^{k} \left(\Omega\tilde{\sigma}^{A}\right) \tilde{\nabla}_{\Xi}^{n-k} \tilde{\nabla}_{A} \tilde{\psi} \\
&= \underbrace{\sum_{k=0}^{n} \binom{n}{k} \tilde{\nabla}_{\Xi}^{k} \left(\Omega\tilde{\sigma}^{A}\right) \tilde{\nabla}_{A} \tilde{\nabla}_{\Xi}^{n-k} \tilde{\psi}}_{\mathrm{V.a}} + \underbrace{\sum_{k=0}^{n} \sum_{i=1}^{n-k} \binom{n}{k} \binom{n-k}{i} \tilde{\nabla}_{\Xi}^{k} \left(\Omega\tilde{\sigma}^{A}\right) \left(\tilde{\nabla}_{\Xi}^{n-k-i} \tilde{\psi}\right) \tilde{\nabla}_{\Xi}^{i-1} \phi_{\Xi A}}_{\mathrm{V.b}}
\end{aligned}$$

we need to use derivatives of the commutator $\phi_{\Omega A}$, that is, we need the relevant terms of (7.66) to evaluate V.b. The relevant terms of V.a are found by analyzing $\tilde{\nabla}_{\Xi}^{k} \left(\Omega\tilde{\sigma}^{A}\right)$. We have

$$\tilde{\nabla}_{\Xi}^{k} \left(\Omega\tilde{\sigma}^{A}\right) = 2\binom{k}{k-2} \tilde{\nabla}_{\Xi}^{k-3} \left(\Xi s^{AB} \sigma_{-} \partial_{\Omega}(\Omega\beta_{B}) - \Xi s^{AC} \partial_{\Omega} \gamma_{BC} \tilde{\sigma}^{B}\right) \tag{7.79}$$

which, after a short calculation, yields

$$\begin{aligned}
\mathrm{V.a}|_{r\to\infty} \simeq\ & n!\tilde{\sigma}^{A} \mathcal{D}_{A} \psi^{(n-2)} - \frac{1}{2} n! s^{AB} \sigma_{-} \sum_{m=0}^{n-4} \beta_{B}^{(m)} \mathcal{D}_{A} \psi^{(n-4-m)} \\
& + \frac{1}{2} n! s^{AB} \tilde{\sigma}^{C} \sum_{m=1}^{n-2} \gamma_{BC}^{(m)} \mathcal{D}_{A} \psi^{(n-2-m)}
\end{aligned} \tag{7.80}$$

and if $n = 2d-2$ there is additionally the term

$$+\frac{1}{4} n! \gamma_{BC}^{(d-2)} \gamma^{AC(d-2)} \tilde{\sigma}^{B} \mathcal{D}_{A} \psi^{(0)}\,. \tag{7.81}$$

The term V.b is more complicated. Using $2\Xi\phi_{\Omega A} = \phi_{\Xi A}$ we have

$$\mathrm{V.b} = 2 \sum_{k=0}^{n} \sum_{i=1}^{n-k} \binom{n}{k} \binom{n-k}{i} \tilde{\nabla}_{\Xi}^{k} \left(\Omega\tilde{\sigma}^{A}\right) \left(\tilde{\nabla}_{\Xi}^{n-k-i} \tilde{\psi}\right) \tilde{\nabla}_{\Xi}^{i-1} (\Xi\phi_{\Omega A})\,. \tag{7.82}$$

and define the abbreviation

$$C(X) := 2 \sum_{k=0}^{n} \sum_{i=1}^{n-k} \binom{n}{k} \binom{n-k}{i} \tilde{\nabla}_{\Xi}^{k} \left(\Omega\tilde{\sigma}^{A}\right) \left(\tilde{\nabla}_{\Xi}^{n-k-i} \tilde{\psi}\right) \tilde{\nabla}_{\Xi}^{i-1} (\Xi X)\,. \tag{7.83}$$

We now look at the terms appearing in $\phi_{\Omega A}$ and the relevant contributions are as follows. We have

$$\tilde{\nabla}_{\Xi}^{i-1} (\Xi \partial_{\Omega}^{2} (\Omega\beta_{A})) = \beta_{A}^{(i)} \frac{i!}{4} (i+2) + O(\Xi) \tag{7.84a}$$

and plugging this in we find

$$\begin{aligned}
C\left(\partial_{\Omega}^{2}(\Omega\beta_{A})[\sigma_{+},\sigma_{-}]\right)\Big|_{r\to\infty} \simeq\ & n! \sum_{i=1}^{n-2} \frac{i+2}{2} \beta_{A}^{(i)} \tilde{\sigma}^{A} [\sigma_{+},\sigma_{-}] \psi^{(n-2-i)} \\
& + \frac{d}{8} n! \beta^{C(d-2)} \gamma_{BC}^{(d-2)} \tilde{\sigma}^{B} [\sigma_{+},\sigma_{-}] \psi^{(0)}\,.
\end{aligned} \tag{7.84b}$$

where the second term is only for $n = 2d - 2$. Next, there is the term

$$\tilde{\nabla}_{\Xi}^{i-1}(\Xi\partial_u\partial_\Omega\gamma_{AB}) = \partial_u\gamma_{AB}^{(i)}\frac{i!}{2} + O(\Xi) \tag{7.85a}$$

which yields

$$C\left(2\Xi\partial_u\partial_\Omega(\gamma_{AB})\tilde{\sigma}^B\sigma_-\right)\Big|_{r\to\infty} \simeq 2n!\sum_{i=1}^{n-2}\gamma_{AB}^{\prime(i)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_-\psi^{(n-2-i)} - \frac{3}{2}n!s^{DC}\gamma_{AC}^{\prime(n/2-1)}\gamma_{BD}^{(n/2-1)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_-\psi^{(0)}\,. \tag{7.85b}$$

with the second term again appearing only if $n = 2d - 2$. The term

$$\tilde{\nabla}_{\Xi}^{i-1}(\Xi\partial_\Omega(\Omega^2\alpha)\partial_\Omega\gamma_{AB}) = \alpha^{(0)}\gamma_{AB}^{(i-2)}\frac{i-2}{i}i! + O(\Xi) \tag{7.86a}$$

gives only

$$C\left(2\Xi\partial_\Omega(\Omega^2\alpha)\partial_\Omega\gamma_{AB}\tilde{\sigma}^B\sigma_-\right)\Big|_{r\to\infty} \simeq n!\sum_{i=1}^{n-2}4\alpha^{(0)}\frac{i-2}{i}\gamma_{AB}^{(i-2)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_-\psi^{(n-2-i)}\,. \tag{7.86b}$$

Using

$$\tilde{\nabla}_{\Xi}^{i-1}(\Xi\partial_A\partial_\Omega(\Omega\beta_B)) = \partial_A\beta_B^{(i-2)}\frac{i!}{2} + O(\Xi) \tag{7.87a}$$

one finds

$$C\left(2\Xi\partial_A\partial_\Omega(\Omega\beta_B)\tilde{\sigma}^B\sigma_-\right)\Big|_{r\to\infty} \simeq n!\sum_{i=1}^{n-2}2\mathcal{D}_A\beta_B^{(i-2)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_-\psi^{(n-2-i)}\,. \tag{7.87b}$$

The contribution from

$$\tilde{\nabla}_{\Xi}^{i-1}(\Xi\partial_{[B}\partial_{|\Omega|}\gamma_{C]A}) = \partial_{[B}\gamma_{C]A}^{(i)}\frac{i!}{2} + O(\Xi) \tag{7.88a}$$

is

$$C\left(\Xi\partial_{[B}\partial_{|\Omega|}\gamma_{C]A})[\tilde{\sigma}^B,\tilde{\sigma}^C]\right)\Big|_{r\to\infty} \simeq n!\sum_{i=1}^{n-2}2\mathcal{D}_{[B}\gamma_{C]A}^{(i)}\tilde{\sigma}^A\tilde{\sigma}^B\tilde{\sigma}^C\psi^{(n-2-i)}\,. \tag{7.88b}$$

The second-to-last term reads

$$\tilde{\nabla}_{\Xi}^{i-1}(\Xi\partial_\Omega\gamma_{BF}\gamma^{FC}\partial_u\gamma_{AC}) = \sum_{l\in\mathbb{N}}\frac{i-l}{2}s^{FC}\gamma_{BF}^{(i-l)}\partial_u\gamma_{AC}^{(l)}(i-1)! + O(\Xi) \tag{7.89a}$$

and contributes

$$C\left(-\Xi\partial_\Omega\gamma_{BF}\gamma^{FC}\partial_u\gamma_{AC}\tilde{\sigma}^B\sigma_-\right)\Big|_{r\to\infty} \simeq -\frac{n!}{2}s^{CD}\gamma_{AC}^{\prime(n/2-1)}\gamma_{BD}^{(n/2-1)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_-\psi^{(0)}\,. \tag{7.89b}$$

if $n = 2d - 2$ and its contribution vanishes otherwise. Finally, we have the term

$$\tilde{\nabla}_{\Xi}^{i-1}(\Xi\partial_\Omega^2\gamma_{AB}) = \gamma_{AB}^{(i+2)}\frac{(i+2)}{4}i! + O(\Xi) \tag{7.90a}$$

with corresponding contribution

$$C\left(2\Xi\partial_\Omega^2\gamma_{AB}\tilde{\sigma}^B\sigma_+\right)\Big|_{r\to\infty} \simeq n!\sum_{i=1}^{n-2}2(i+2)\gamma_{AB}^{(i+2)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_+\psi^{(n-2-i)}-\frac{n!}{2}\sum_{i+j+k=n}s^{CD}j\,\gamma_{AC}^{(j)}\gamma_{BD}^{(k)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_+\psi^{(i)}\,. \tag{7.90b}$$

with $\frac{d-2}{2}n!\sigma_-\tilde{\sigma}^B\sigma_+\beta^{B(d-2)}\gamma_{AB}^{(d-2)}\psi^{(0)}$ as a third term if $n = 2d-2$. All the other terms appearing in $\phi_{\Omega A}$ give only vanishing contributions in the end and we do not write their explicit form here. Taking the prefactor $\frac{1}{8}$ appearing in $\phi_{\Omega A}$ into account we can sum all the contributions up thereby obtaining

$$\begin{aligned}\mathrm{V.b}|_{r\to\infty} \simeq\ &\frac{n!}{8}\tilde{\sigma}^A\sum_{i=1}^{n-2}\Big[\frac{i+2}{2}\beta_A^{(i)}[\sigma_+,\sigma_-]+2\gamma_{AB}^{\prime(i)}\tilde{\sigma}^B\sigma_-+4\alpha^{(0)}\frac{i-2}{i}\gamma_{AB}^{(i-2)}\tilde{\sigma}^B\sigma_-+2\mathcal{D}_A\beta_B^{(i-2)}\tilde{\sigma}^B\sigma_-\\&+2\mathcal{D}_{[B}\gamma_{C]A}^{(i)}\tilde{\sigma}^B\tilde{\sigma}^C+2(i+2)\gamma_{AB}^{(i+2)}\tilde{\sigma}^B\sigma_+\Big]\psi^{(n-2-i)}-\frac{n!}{16}\sum_{i+j+k=n}s^{CD}j\,\gamma_{AC}^{(j)}\gamma_{BD}^{(k)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_+\psi^{(i)}\\&+\Big\{-\frac{3}{16}n!s^{DC}\gamma_{AC}^{\prime(n/2-1)}\gamma_{BD}^{(n/2-1)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_-\psi^{(0)}-\frac{n!}{16}s^{CD}\gamma_{AC}^{\prime(n/2-1)}\gamma_{BD}^{(n/2-1)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_-\psi^{(0)}\\&+\frac{d}{64}n!\beta^{C(d-2)}\gamma_{BC}^{(d-2)}\tilde{\sigma}^B[\sigma_+,\sigma_-]\psi^{(0)}+\frac{d-2}{2}n!\sigma_-\tilde{\sigma}^B\sigma_+\beta^{A(d-2)}\gamma_{AB}^{(d-2)}\psi^{(0)}\Big\}_{\text{if }n=2d-2}\,.\end{aligned} \tag{7.91}$$

where the terms in the curly bracket are there only if $n = 2d-2$. Using the identities

$$\begin{aligned}&\gamma_{AB}\sigma^A\sigma^B = \frac{1}{2}\gamma_{AB}\sigma^A\sigma^B+\frac{1}{2}\gamma_{BA}\sigma^B\sigma^A = \frac{1}{2}\gamma_{AB}\left\{\sigma^A,\sigma^B\right\} = \gamma_{AB}s^{AB}\,,\ \text{and}\\&\mathcal{D}_{[B}\gamma_{C]A}\sigma^A\sigma^B\sigma^C = \left[\mathcal{D}^B\gamma_{AB}-\mathcal{D}_A\gamma\right]\sigma^A\end{aligned}$$

we can rewrite this to

$$\begin{aligned}\mathrm{V.b}|_{r\to\infty} \simeq\ &\frac{n!}{8}\tilde{\sigma}^A\sum_{i=1}^{n-2}\Big\{\frac{i+2}{2}\beta_A^{(i)}[\sigma_+,\sigma_-]+2\gamma_{AB}^{\prime(i)}\tilde{\sigma}^B\sigma_-+4\alpha^{(0)}\frac{i-2}{i}\gamma_{AB}^{(i-2)}\tilde{\sigma}^B\sigma_-+2\mathcal{D}_A\beta_B^{(i-2)}\tilde{\sigma}^B\sigma_-\\&+2\left[\mathcal{D}^B\gamma_{AB}^{(i)}-\mathcal{D}_A\gamma^{(i)}\right]+2(i+2)\gamma_{AB}^{(i+2)}\tilde{\sigma}^B\sigma_+\Big\}\psi^{(n-2-i)}-\frac{n!}{16}\sum_{i+j+k=n}s^{CD}j\,\gamma_{AC}^{(j)}\gamma_{BD}^{(k)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_+\psi^{(i)}\\&+\Big\{-\frac{n!}{4}s^{DC}\gamma_{AC}^{\prime(n/2-1)}\gamma_{BD}^{(n/2-1)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_-\psi^{(0)}+\frac{d}{64}n!\beta^{C(d-2)}\gamma_{BC}^{(d-2)}\tilde{\sigma}^B[\sigma_+,\sigma_-]\psi^{(0)}\\&+\frac{d-2}{2}n!\sigma_-\tilde{\sigma}^B\sigma_+\beta^{A(d-2)}\gamma_{AB}^{(d-2)}\psi^{(0)}\Big\}_{\text{if }n=2d-2}\,.\end{aligned} \tag{7.92}$$

where for the terms in the curly bracket holds the same as above. Now, adding (7.80) and (7.92) yields the full expression for V. The sixth term of the Witten equation is

$$\mathrm{VI} := -\frac{d}{2}\sum_{k=0}^{n}\binom{n}{k}\tilde{\nabla}_\Xi^k\sigma_+\tilde{\nabla}_\Xi^{n-k}\tilde{\psi}$$

and, after taking the limit, we find

$$\mathrm{VI}|_{r\to\infty} \simeq -\frac{d}{2}n!\sigma_+\psi^{(n)} + \frac{d}{4}n!\sum_{m=2}^{n}\beta_A^{(m-2)}\tilde{\sigma}^A\psi^{(n-m)}\,. \tag{7.93}$$

with an additional term $-\frac{d}{8}n!\frac{d}{d-2}\beta^{C(d-2)}\gamma_{AC}^{(d-2)}\tilde{\sigma}^A\psi^{(0)}$ if $n = 2d-2$. The second-to-last term is

$$\mathrm{VII} := \left(1-\frac{d}{2}\right)\sum_{k=0}^{n}\binom{n}{k}\tilde{\nabla}_\Xi^k\left(\alpha\sigma_-\right)\tilde{\nabla}_\Xi^{n-k}(\Omega^2\tilde{\psi})\,.$$

It can be rewritten as

$$\mathrm{VII} = \left(1-\frac{d}{2}\right)\sum_{k=0}^{n}\binom{n}{k}\left(\partial_\Xi^k\alpha\right)\sigma_-\tilde{\nabla}_\Xi^{n-k}(\Omega^2\tilde{\psi})$$

and thus

$$\mathrm{VII}|_{r\to\infty} \simeq \left(1-\frac{d}{2}\right)n!\sigma_-\sum_{m=0}^{n-4}\alpha^{(m)}\psi^{(n-4-m)}\,. \tag{7.94}$$

Finally, the eighth and last term is

$$\mathrm{VIII} := -\frac{d}{2}\sum_{k=0}^{n}\binom{n}{k}\tilde{\nabla}_\Xi^k\left(\Omega\beta_A\tilde{\sigma}^A\right)\tilde{\nabla}_\Xi^{n-k}\tilde{\psi}$$

which is equal to

$$\mathrm{VIII} = -\frac{d}{2}\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}\partial_\Xi^j\left(\Omega\beta_A\right)\left(\tilde{\nabla}_\Xi^{k-j}\tilde{\sigma}^A\right)\tilde{\nabla}_\Xi^{n-k}\tilde{\psi}\,.$$

Therefore, the result for the last term is

$$\mathrm{VIII}|_{r\to\infty} \simeq -\frac{d}{2}n!\sum_{m=0}^{n-2}\beta_A^{(m)}\tilde{\sigma}^A\psi^{(n-2-m)} \tag{7.95}$$

and if $n = 2d - 2$ one has to add the term $-\frac{d}{4} n! \beta^{B(d-2)} \gamma_{AB}^{(d-2)} \tilde{\sigma}^A \psi^{(0)}$. Now we have found all contributing terms. Adding the eight terms together and dividing by $n!$ yields

$$
\begin{aligned}
0 &= \frac{n}{2}\sigma_+\psi^{(n)} - \tilde{\sigma}^A \sum_{k=1}^{n} \frac{n-k}{4}\beta_A^{(k-2)}\psi^{(n-k)} + n\sigma_-\psi^{(n)} - \sigma_- \sum_{k=1}^{n-4}\frac{k}{2}\alpha^{(n-4-k)}\psi^{(k)} \\
&+ \sum_{m=0}^{n-3}\frac{n-m-2}{2}\beta_A^{(m)}\tilde{\sigma}^A\psi^{(n-m-2)} - \sum_{m=0}^{n-4}\beta^{A(n-4-m)}\sigma_-\mathcal{D}_A\psi^{(m)} \\
&+ \tilde{\sigma}^A\mathcal{D}_A\psi^{(n-2)} - \frac{1}{2}s^{AB}\sigma_-\sum_{m=0}^{n-4}\beta_B^{(m)}\mathcal{D}_A\psi^{(n-4-m)} + \frac{1}{2}s^{AB}\tilde{\sigma}^C\sum_{m=1}^{n-2}\gamma_{BC}^{(m)}\mathcal{D}_A\psi^{(n-2-m)} \\
&+ \frac{1}{8}\tilde{\sigma}^A\sum_{i=1}^{n-2}\left\{\frac{i+2}{2}\beta_A^{(i)}[\sigma_+,\sigma_-] + 2\gamma_{AB}^{\prime(i)}\tilde{\sigma}^B\sigma_- + 4\alpha^{(0)}\frac{i-2}{i}\gamma_{AB}^{(i-2)}\tilde{\sigma}^B\sigma_- + 2\mathcal{D}_A\beta_B^{(i-2)}\tilde{\sigma}^B\sigma_-\right. \\
&\left. +2\left[\mathcal{D}^B\gamma_{AB}^{(i)} - \mathcal{D}_A\gamma^{(i)}\right] + 2(i+2)\gamma_{AB}^{(i+2)}\tilde{\sigma}^B\sigma_+\right\}\psi^{(n-2-i)} - \frac{1}{16}\sum_{i+j+k=n} s^{CD} j\,\gamma_{AC}^{(j)}\gamma_{BD}^{(k)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_+\psi^{(i)} \\
&- \frac{d}{2}\sigma_+\psi^{(n)} + \frac{d}{4}\sum_{m=2}^{n}\beta_A^{(m-2)}\tilde{\sigma}^A\psi^{(n-m)} + \left(1-\frac{d}{2}\right)\sigma_-\sum_{m=0}^{n-4}\alpha^{(m)}\psi^{(n-4-m)} - \frac{d}{2}\sum_{m=0}^{n-2}\beta_A^{(m)}\tilde{\sigma}^A\psi^{(n-2-m)}
\end{aligned}
\tag{7.96}
$$

for $1 \leq n < 2d - 2$ and if $n = 2d - 2$ there are the following terms added to the right-hand side.

$$
\begin{aligned}
&\frac{n-2}{4}\gamma^{AB(n/2-1)}\beta_B^{(n/2-1)}\sigma_-\mathcal{D}_A\psi^{(0)} + \frac{1}{4}\gamma_{BC}^{(d-2)}\gamma^{AC(d-2)}\tilde{\sigma}^B\mathcal{D}_A\psi^{(0)} + \frac{d}{64}\beta^{C(d-2)}\gamma_{BC}^{(d-2)}\tilde{\sigma}^B[\sigma_+,\sigma_-]\psi^{(0)} \\
&- \frac{1}{4}s^{DC}\gamma_{AC}^{\prime(n/2-1)}\gamma_{BD}^{(n/2-1)}\tilde{\sigma}^A\tilde{\sigma}^B\sigma_-\psi^{(0)} - \frac{d}{8}\frac{d}{d-2}\beta^{C(d-2)}\gamma_{AC}^{(d-2)}\tilde{\sigma}^A\psi^{(0)} - \frac{d}{4}\beta^{B(d-2)}\gamma_{AB}^{(d-2)}\tilde{\sigma}^A\psi^{(0)} \\
&+ \frac{d-2}{2}\sigma_-\tilde{\sigma}^B\sigma_+\beta^{A(d-2)}\gamma_{AB}^{(d-2)}\psi^{(0)}
\end{aligned}
\tag{7.97}
$$

This is the explicit expression of the Witten equation at order $r^{-n/2}$ we wanted to find.

7.8 Proof Lemma 7.3— Step 4: Recursion Formula Spinor

We know apply the projectors $P_\pm$ to the Witten equation to find recursion relations for the components ψ_+ and ψ_-. Applying $P_+ = \frac{1}{2}\sigma_+ \cdot \sigma_-$ to (7.96) yields for $1 \le n < 2d-2$

$$
\begin{aligned}
0 = &\frac{n}{2}\sqrt{2}\psi_-^{(n)} - \Gamma^A \sum_{k=1}^{n} \frac{n-k}{4}\beta_A^{(k-2)}\psi_+^{(n-k)} + \sum_{m=0}^{n-3}\frac{n-m-2}{2}\beta_A^{(m)}\Gamma^A\psi_+^{(n-m-2)} \\
&+ \Gamma^A\mathcal{D}_A\psi_+^{(n-2)} + \frac{1}{2}s^{AB}\Gamma^C\sum_{m=1}^{n-2}\gamma_{BC}^{(m)}\mathcal{D}_A\psi_+^{(n-2-m)} - \frac{d}{2}\sum_{m=0}^{n-2}\beta_A^{(m)}\Gamma^A\psi_+^{(n-2-m)} \\
&+ \frac{1}{8}\Gamma^A\sum_{i=1}^{n-2}\left\{\frac{i+2}{2}\beta_A^{(i)}\psi_+^{(n-2-i)} + 2\left[\mathcal{D}^B\gamma_{AB}^{(i)} - \mathcal{D}_A\gamma^{(i)}\right]\psi_+^{(n-2-i)} + 2(i+2)\gamma_{AB}^{(i+2)}\Gamma^B\sqrt{2}\psi_-^{(n-2-i)}\right\} \\
&- \frac{1}{16}\sum_{i+j+k=n} s^{CD} j\,\gamma_{AC}^{(j)}\gamma_{BD}^{(k)}\Gamma^A\Gamma^B\sqrt{2}\psi_-^{(i)} - \frac{d}{2}\sqrt{2}\psi_-^{(n)} + \frac{d}{4}\sum_{m=2}^{n}\beta_A^{(m-2)}\Gamma^A\psi_+^{(n-m)}
\end{aligned}
\tag{7.98}
$$

where the relations in (7.26) were used. This can be solved for $\psi_-^{(n)}$ such that

$$
\begin{aligned}
\psi_-^{(n)} = &\frac{\sqrt{2}}{d-n}\left\{-\Gamma^A\sum_{k=1}^{n}\frac{n-k}{4}\beta_A^{(k-2)}\psi_+^{(n-k)} + \sum_{m=0}^{n-3}\frac{n-m-2}{2}\beta_A^{(m)}\Gamma^A\psi_+^{(n-m-2)}\right. \\
&+ \Gamma^A\mathcal{D}_A\psi_+^{(n-2)} + \frac{1}{2}s^{AB}\Gamma^C\sum_{m=1}^{n-2}\gamma_{BC}^{(m)}\mathcal{D}_A\psi_+^{(n-2-m)} \\
&+ \frac{1}{8}\Gamma^A\sum_{i=1}^{n-2}\left\{\frac{i+2}{2}\beta_A^{(i)}\psi_+^{(n-2-i)} + 2\left[\mathcal{D}^B\gamma_{AB}^{(i)} - \mathcal{D}_A\gamma^{(i)}\right]\psi_+^{(n-2-i)} + 2(i+2)\gamma_{AB}^{(i+2)}\Gamma^B\sqrt{2}\psi_-^{(n-2-i)}\right\} \\
&\left.-\frac{1}{16}\sum_{i+j+k=n} s^{CD} j\,\gamma_{AC}^{(j)}\gamma_{BD}^{(k)}\Gamma^A\Gamma^B\sqrt{2}\psi_-^{(i)} + \frac{d}{4}\sum_{m=2}^{n}\beta_A^{(m-2)}\Gamma^A\psi_+^{(n-m)} - \frac{d}{2}\sum_{m=0}^{n-2}\beta_A^{(m)}\Gamma^A\psi_+^{(n-2-m)}\right\}.
\end{aligned}
\tag{7.99}
$$

Assuming that $\psi_+^{(0)} = \epsilon$ and $\psi_-^{(0)} = 0$ we immediately see that

$$\psi_-^{(1)} = 0 \tag{7.100}$$

and

$$\psi_-^{(2)} = \frac{\sqrt{2}}{(d-2)}\Gamma^A\mathcal{D}_A\psi_+^{(0)} \tag{7.101}$$

For $n = 2d-2$ are additionally the terms

$$
\begin{aligned}
&\frac{\sqrt{2}}{d-n}\left\{\frac{1}{4}\gamma_{BC}^{(d-2)}\gamma^{AC(d-2)}\Gamma^B\mathcal{D}_A\psi_+^{(0)} + \frac{d}{64}\beta^{C(d-2)}\gamma_{BC}^{(d-2)}\Gamma^B\psi_+^{(0)}\right. \\
&\left.-\frac{d}{8}\frac{d}{d-2}\beta^{C(d-2)}\gamma_{AC}^{(d-2)}\Gamma^A\psi_+^{(0)} - \frac{d}{4}\beta^{B(d-2)}\gamma_{AB}^{(d-2)}\Gamma^A\psi_+^{(0)}\right\}
\end{aligned}
\tag{7.102}
$$

which have to be added on the right-hand side.

Applying instead P_- to (7.96) gives

$$
\begin{aligned}
0 = &-\frac{1}{2}\beta_A^{(n-4)}\Gamma^A\psi_-^{(2)} + n\sqrt{2}\psi_+^{(n)} - \sqrt{2}\frac{n-4}{2}\alpha^{(0)}\psi_+^{(n-4)} + \beta_A^{(n-4)}\Gamma^A\psi_-^{(2)} - \beta^{A(n-4)}\sqrt{2}\mathcal{D}_A\psi_+^{(0)} \\
&+ \Gamma^A\mathcal{D}_A\psi_-^{(n-2)} - \frac{1}{2}\sqrt{2}\beta^{A(n-4)}\mathcal{D}_A\psi_+^{(0)} + \frac{1}{2}\Gamma^C\gamma_{BC}^{(n-4)}\mathcal{D}^B\psi_-^{(2)} \\
&+ \frac{1}{8}\Gamma^A\left\{-\frac{n-2}{2}\beta_A^{(n-4)}\psi_-^{(2)} + 2\gamma_{AB}^{\prime(n-2)}\Gamma^B\sqrt{2}\psi_+^{(0)} + 4\alpha^{(0)}\frac{n-4}{n-2}\gamma_{AB}^{(n-4)}\Gamma^B\sqrt{2}\psi_+^{(0)} + 2\mathcal{D}_A\beta_B^{(n-4)}\Gamma^B\sqrt{2}\psi_+^{(0)}\right. \\
&\left.+2\left[\mathcal{D}^B\gamma_{AB}^{(n-4)} - \mathcal{D}_A\gamma^{(n-4)}\right]\psi_-^{(2)}\right\} + \frac{d}{4}\beta_A^{(n-4)}\Gamma^A\psi_-^{(2)} \\
&+ \left(1-\frac{d}{2}\right)\sqrt{2}\alpha^{(n-4)}\psi_+^{(0)} + \left(1-\frac{d}{2}\right)\sqrt{2}\alpha^{(0)}\psi_+^{(n-4)} - \frac{d}{2}\beta_A^{(n-4)}\Gamma^A\psi_-^{(2)}
\end{aligned}
\tag{7.103}
$$

where again there are additional terms, namely

$$
+\frac{n-2}{4}\gamma^{AB(n/2-1)}\beta_B^{(n/2-1)}\sqrt{2}\mathcal{D}_A\psi_+^{(0)} - \frac{1}{4}s^{DC}\gamma_{AC}^{\prime(n/2-1)}\gamma_{BD}^{(n/2-1)}\Gamma^A\Gamma^B\sqrt{2}\psi_+^{(0)} + \frac{d-2}{4}\Gamma^B\beta^{A(d-2)}\gamma_{AB}^{(d-2)}\psi_-^{(0)}\,,
\tag{7.104}
$$

if $n = 2d-2$. We thus find for $\psi_-^{(n)}$ the expression

$$
\begin{aligned}
\psi_-^{(n)} = \frac{\sqrt{2}}{d-n}&\left\{\Gamma^A\mathcal{D}_A\psi_+^{(n-2)} + \frac{1}{2}\gamma_{AB}^{(n-2)}\Gamma^B\mathcal{D}^A\psi_+^{(0)} + \frac{n-4d}{16}\beta_A^{(n-2)}\Gamma^A\psi_+^{(0)}\right. \\
&\left.+\frac{1}{4}\Gamma^A\left[\mathcal{D}^B\gamma_{AB}^{(n-2)} - \mathcal{D}_A\gamma^{(n-2)}\right]\psi_+^{(0)} + \frac{n-2}{\sqrt{2}}\gamma^{(n-2)}\psi_-^{(2)}\right\}.
\end{aligned}
\tag{7.105}
$$

for $1 \le n < 2d-2$ while for $n = 2d-2$ there are also the terms

$$
+\frac{1}{4}\gamma_{BC}^{(d-2)}\gamma^{AC(d-2)}\Gamma^B\mathcal{D}_A\psi_+^{(0)} + \frac{30d-23d^2}{64(d-2)}\beta^{C(d-2)}\gamma_{BC}^{(d-2)}\Gamma^B\psi_+^{(0)} + \frac{n-2}{16\sqrt{2}}\gamma_{AB}^{(n/2-1)}\gamma^{(n/2-1)AB}\psi_-^{(2)}\,.
\tag{7.106}
$$

For $\psi_+^{(n)}$ we find for $1 \le n < 2d-2$

$$
\begin{aligned}
\psi_+^{(n)} = \frac{-1}{\sqrt{2}n}&\left\{\frac{10-n-2d}{16}\beta_A^{(n-4)}\Gamma^A\psi_-^{(2)} - \frac{6-d-n}{\sqrt{2}}\alpha^{(0)}\psi_+^{(n-4)} + \beta_A^{(n-4)}\Gamma^A\psi_-^{(2)} - \beta^{A(n-4)}\sqrt{2}\mathcal{D}_A\psi_+^{(0)}\right. \\
&+ \Gamma^A\mathcal{D}_A\psi_-^{(n-2)} - \frac{1}{\sqrt{2}}\beta^{A(n-4)}\mathcal{D}_A\psi_+^{(0)} + \frac{1}{2}\Gamma^C\gamma_{BC}^{(n-4)}\mathcal{D}^B\psi_-^{(2)} + \frac{1}{4}\left[\mathcal{D}^B\gamma_{AB}^{(n-4)} - \mathcal{D}_A\gamma^{(n-4)}\right]\Gamma^A\psi_-^{(2)} \\
&+ \frac{1}{2\sqrt{2}}\gamma^{\prime(n-2)}\psi_+^{(0)} + \frac{1}{\sqrt{2}}\alpha^{(0)}\frac{n-4}{n-2}\gamma^{(n-4)}\psi_+^{(0)} + \frac{1}{2\sqrt{2}}\mathcal{D}_A\beta_B^{(n-4)}\Gamma^A\Gamma^B\psi_+^{(0)} \\
&\left.+\left(1-\frac{d}{2}\right)\sqrt{2}\alpha^{(n-4)}\psi_+^{(0)}\right\}
\end{aligned}
\tag{7.107}
$$

and for $n = 2d-2$ the terms

$$
+\frac{n-2}{2\sqrt{2}}\gamma^{AB(n/2-1)}\beta_B^{(n/2-1)}\mathcal{D}_A\psi_+^{(0)} - \frac{1}{2\sqrt{2}}\gamma_{AB}^{\prime(n/2-1)}\gamma^{AB(n/2-1)}\psi_+^{(0)} + \frac{d-2}{4}\Gamma^B\beta^{A(d-2)}\gamma_{AB}^{(d-2)}\psi_-^{(0)}
\tag{7.108}
$$

are added. This yields the expressions stated in 7.3. □

7.9 Existence of Integral

We can now go back to the integral (7.46) and show that the limit exists by deriving an expression for the coefficients $Q^{(n)}$ of the asymptotic expansion of Q. We have

$$\begin{aligned} Q(\tilde{e}^+, \tilde{e}^-) &= \Re\left(\bar{\psi}\tilde{e}^- \cdot \nabla_{\tilde{e}^+}\psi - \bar{\psi}\tilde{e}^+ \cdot \nabla_{\tilde{e}^-}\psi\right) = \Re\left(\bar{\psi}\tilde{e}^-_b\tilde{\sigma}^b\tilde{e}^{+a}\nabla_a\psi - \bar{\psi}\tilde{e}^+_b\tilde{\sigma}^b\tilde{e}^{-a}\nabla_a\psi\right) \\ &= \Re\left(\bar{\psi}\tilde{e}^-_\Omega\tilde{\sigma}^\Omega\tilde{e}^{+\Omega}\nabla_\Omega\psi + \bar{\psi}\tilde{e}^-_u\tilde{\sigma}^u\tilde{e}^{+\Omega}\nabla_\Omega\psi + \bar{\psi}\tilde{e}^-_A\tilde{\sigma}^A\tilde{e}^{+\Omega}\nabla_\Omega\psi - \bar{\psi}\tilde{e}^+_u\tilde{\sigma}^u\tilde{e}^{-\Omega}\nabla_\Omega\psi - \bar{\psi}\tilde{e}^+_u\tilde{\sigma}^u\tilde{e}^{-u}\nabla_u\psi\right) . \end{aligned}$$

Substituting the components of the tetrad we chose, i.e., (7.9) and (7.10), yields

$$Q(\tilde{e}^+, \tilde{e}^-) = \Re\left(\bar{\psi}\tilde{\sigma}^\Omega\nabla_\Omega\psi - 2\bar{\psi}\Omega^2\alpha\tilde{\sigma}^u\nabla_\Omega\psi - \Omega\beta_A\bar{\psi}\tilde{\sigma}^A\nabla_\Omega\psi - \bar{\psi}\tilde{\sigma}^u\nabla_u\psi\right) . \tag{7.109}$$

Replacing $\tilde{\sigma}^\Omega$ and $\tilde{\sigma}^u$ with the corresponding components in (7.12) and using the physical version of (7.59) to replace ∇_u, found by repeating the calculation leading to (7.59) for ∇_u, gives

$$\begin{aligned} Q(\tilde{e}^+, \tilde{e}^-) &= \Re\Big(\bar{\psi}\left[\sigma_+ + \Omega^2\alpha\sigma_- + \Omega\beta_A\tilde{\sigma}^A\right]\nabla_\Omega\psi - 2\bar{\psi}\alpha\Omega^2\sigma_-\nabla_\Omega\psi - \Omega\beta_A\bar{\psi}\tilde{\sigma}^A\nabla_\Omega\psi \\ &\quad -\bar{\psi}\sigma_-\left[\left(-2\Omega^2\alpha + 2 - \Omega^2\beta^A\beta_A\right)\nabla_\Omega - 2\Omega\beta^A\nabla_A\right]\psi\Big) \\ &= \Re\left(\bar{\psi}\sigma_+\nabla_\Omega\psi + \bar{\psi}\sigma_-\left[\left(\Omega^2\alpha - 2 + \Omega^2\beta^A\beta_A\right)\nabla_\Omega + 2\Omega\beta^A\nabla_A\right]\psi - \Omega\beta_A\bar{\psi}\tilde{\sigma}^A\nabla_\Omega\psi\right) \\ &\simeq \Re\left(\bar{\psi}\sigma_+\nabla_\Omega\psi + \bar{\psi}\sigma_-\left(\Omega^2\alpha - 2\right)\nabla_\Omega\psi\right) . \end{aligned}$$

Looking at the nth order we find

$$Q^{(n)}(\tilde{e}^+, \tilde{e}^-) = \sqrt{2}\Re\left(\frac{n}{2}\left\langle\psi_-^{(2)}, \psi_-^{(n)}\right\rangle - (n+2)\left\langle\psi_+^{(0)}, \psi_+^{(n+2)}\right\rangle + \frac{n-2}{2}\alpha^{(0)}\left\langle\psi_+^{(0)}, \psi_+^{(n-2)}\right\rangle\right) . \tag{7.110}$$

To bring this into a more suitable form we use the Killing-spinor equation

$$\mathcal{D}_X\epsilon = \frac{i\lambda}{2}X \cdot \epsilon \Leftrightarrow X^a\mathcal{D}_a\epsilon = \frac{i\lambda}{2}X^a\Gamma_a\epsilon . \tag{7.111}$$

Thus, we have

$$\Gamma^A\mathcal{D}_A\epsilon = \frac{i\lambda}{2}(d-2)\epsilon$$

and multiplying this by $\Gamma^B\mathcal{D}_B$ yields

$$\Gamma^B\mathcal{D}_B\Gamma^A\mathcal{D}_A\epsilon = \frac{i\lambda}{2}(d-2)\Gamma^B\mathcal{D}_B\epsilon = -\frac{(d-2)^2\lambda^2}{4}\epsilon = -\frac{1}{2}(d-2)^2\alpha^{(0)}\psi_+^{(0)} ,$$

and hence

$$\Gamma^A\Gamma^B\mathcal{D}_A\mathcal{D}_B\epsilon = -\frac{1}{2}(d-2)^2\alpha^{(0)}\psi_+^{(0)} . \tag{7.112}$$

This can be used to rewrite the last term in (7.110), where $\epsilon = \psi_+^{(0)}$ by assumption, as

$$\begin{aligned}\alpha^{(0)}\left\langle \psi_+^{(0)}, \psi_+^{(n-2)}\right\rangle &= \left\langle \alpha^{(0)}\psi_+^{(0)}, \psi_+^{(n-2)}\right\rangle = \left\langle -\frac{2}{(d-2)^2}\Gamma^A\Gamma^B\mathcal{D}_A\mathcal{D}_B\epsilon, \psi_+^{(n-2)}\right\rangle \\ &= -\frac{2}{(d-2)^2}\left\langle \Gamma^A\Gamma^B\mathcal{D}_A\mathcal{D}_B\epsilon, \psi_+^{(n-2)}\right\rangle . \end{aligned} \tag{7.113}$$

Therefore,

$$Q^{(n)}(\tilde{e}^+,\tilde{e}^-) = \sqrt{2}\Re\left(\frac{n}{2}\left\langle \psi_-^{(2)}, \psi_-^{(n)}\right\rangle - (n+2)\left\langle \psi_+^{(0)}, \psi_+^{(n+2)}\right\rangle - \frac{n-2}{(d-2)^2}\left\langle \Gamma^A\Gamma^B\mathcal{D}_A\mathcal{D}_B\epsilon, \psi_+^{(n-2)}\right\rangle\right) . \tag{7.114}$$

With $\psi_-^{(2)} = i\lambda/(\sqrt{2})\epsilon$ this is

$$Q^{(n)}(\tilde{e}^+,\tilde{e}^-) = \sqrt{2}\Re\left(\frac{i\lambda n}{2\sqrt{2}}\left\langle \epsilon, \psi_-^{(n)}\right\rangle - (n+2)\left\langle \epsilon, \psi_+^{(n+2)}\right\rangle - \frac{n-2}{(d-2)^2}\left\langle \Gamma^A\Gamma^B\mathcal{D}_A\mathcal{D}_B\epsilon, \psi_+^{(n-2)}\right\rangle\right) . \tag{7.115}$$

In the last term, we can move one derivative to the second factor which produces a total derivative,

$$Q^{(n)} = \sqrt{2}\Re\left(\underbrace{\frac{i\lambda n}{2\sqrt{2}}\left\langle \epsilon, \psi_-^{(n)}\right\rangle}_{\mathrm{I}} - \underbrace{(n+2)\left\langle \epsilon, \psi_+^{(n+2)}\right\rangle}_{\mathrm{II}} - \underbrace{\frac{n-2}{(d-2)^2}\Gamma^A\Gamma^B\left\langle \mathcal{D}_B\epsilon, \mathcal{D}_A\psi_+^{(n-2)}\right\rangle}_{\mathrm{III}} + \mathcal{D}_A\omega^A\right) , \tag{7.116}$$

where ω^A is some function. It turns out that, in the end, ω^A vanishes independently of its precise form so we do not explicitly write it here to keep the expressions shorter. It is only important that $\mathcal{D}_A\omega^A$ is a total derivative. Note that in the following calculations many total derivatives will appear and we will always add them to $\mathcal{D}_A\omega^A$ without explicitly mentioning it every time and hence the exact form of ω^A might change from line to line.

Let $\langle \epsilon, \epsilon\rangle = |\epsilon|^2 \in \mathbb{R}$, $1 \le n < 2d-4$, and recall that $\psi_+^{(0)} = \epsilon$ and $\psi_-^{(2)} = i\lambda/(\sqrt{2})\epsilon$. We will now use the recursion relations in Lemma 7.3 to show that (7.116) can be brought in a form such that the integral (7.46) exists and is equal to the Bondi mass. Since the expressions are again rather lengthy we consider the three terms I-III of (7.116) independently. For the term I we need

$$\begin{aligned}\frac{d-n}{\sqrt{2}}\left\langle \epsilon, \psi_-^{(n)}\right\rangle = \Big\{&\Gamma^A\left\langle \epsilon, \mathcal{D}_A\psi_+^{(n-2)}\right\rangle + \frac{n-4d}{16}\beta_A^{(n-2)}\Gamma^A|\epsilon|^2 + \frac{1}{4}(\mathcal{D}^B\gamma_{AB}^{(n-2)} - \mathcal{D}_A\gamma^{(n-2)})\Gamma^A|\epsilon|^2 \\ &+\frac{1}{2}\gamma_{AB}^{(n-2)}\Gamma^B\left\langle \epsilon, \mathcal{D}^A\psi_+^{(0)}\right\rangle\Big\}\end{aligned}$$

Since only the real part of term I will enter we only have one relevant term that potentially could be of importance,

$$\Re\left\{\frac{i\lambda n}{2(d-n)}\left\langle \epsilon, \psi_-^{(n)}\right\rangle\right\} = \Re\left\{\frac{i\lambda n}{2(d-n)}\Gamma^A\left\langle \epsilon, \mathcal{D}_A\psi_+^{(n-2)}\right\rangle\right\} . \tag{7.117}$$

A similar argument can be used to significantly reduce the length of the third term. For this note that, since ϵ is a Killing spinor, we have

$$\Gamma^A \mathcal{D}_A \epsilon = \frac{i\lambda(d-2)}{2}\epsilon \tag{7.118}$$

and thus all terms of the form $\left\langle \epsilon, \Gamma^A \mathcal{D}_A \epsilon \right\rangle$ are purely imaginary, thus not contributing to $Q^{(n)}$, while terms

$$\left\langle \Gamma^A \mathcal{D}_A \epsilon, \Gamma^B \mathcal{D}_B \epsilon \right\rangle = -\frac{(d-2)^2\lambda^2}{4} |\epsilon|^2 \tag{7.119}$$

are real and matter. Thus we have,

$$\begin{aligned}
&-(n-2)\Re\left\{\Gamma^A\Gamma^B\left\langle \mathcal{D}_B\epsilon, \mathcal{D}_A\psi_+^{(n-2)}\right\rangle\right\} = \\
&\qquad \frac{1}{\sqrt{2}}\Gamma^A\Gamma^B\Gamma^C\Re\left\langle \mathcal{D}_B\epsilon, \mathcal{D}_A\mathcal{D}_C\psi_-^{(n-4)}\right\rangle - \frac{8-d-n}{2}\alpha^{(0)}\Gamma^A\Gamma^B\Re\left\langle \mathcal{D}_B\epsilon, \mathcal{D}_A\psi_+^{(n-6)}\right\rangle \\
&\qquad - \frac{(d-2)^2\lambda^2}{16}\gamma'^{(n-4)}|\epsilon|^2 - \frac{(d-2)^2\lambda^2(n-6)}{8(n-4)}\alpha^{(0)}\gamma^{(n-6)}|\epsilon|^2 + \frac{(d-2)^3\lambda^2}{8}\alpha^{(n-6)}|\epsilon|^2 \\
&\qquad - \frac{(d-2)^2\lambda^2}{16}\Gamma^A\Gamma^B\mathcal{D}_A\beta_B^{(n-6)}|\epsilon|^2 + O(\text{total derivatives})\,.
\end{aligned}$$

We now consider term II which includes the scalar product

$$\begin{aligned}
-(n+2)\Re\left\langle \epsilon, \psi_+^{(n+2)}\right\rangle &= \frac{1}{\sqrt{2}}\Gamma^A\Re\left\langle \epsilon, \mathcal{D}_A\psi_-^{(n)}\right\rangle - \frac{4-d-n}{2}\alpha^{(0)}\Re\left\langle \epsilon, \psi_+^{(n-2)}\right\rangle + \frac{1}{4}\mathcal{D}_B\beta_A^{(n-2)}\Gamma^A\Gamma^B|\epsilon|^2 \\
&\quad + \frac{1}{4}\gamma'^{(n)}|\epsilon|^2 + \frac{2-d}{2}\alpha^{(n-2)}|\epsilon|^2 + \frac{4n-8-\sqrt{2}n}{4\sqrt{2}n}\alpha^{(0)}\gamma^{(n-2)}|\epsilon|^2\,.
\end{aligned}$$

We can now add all three terms together. Using the results of Lemma 6.1 and including all total derivatives in $\mathcal{D}^A\omega_A^{(n)}$, we see that, for $1 \le n < 2d-4$, we have

$$Q^{(n)} = \mathcal{D}^A\omega_A^{(n)}\,. \tag{7.120}$$

Thus, for this range of n, $Q^{(n)}$ is equal to a total derivative and will therefore vanish in the integral. Now consider $n = 2d-4$. The above results for the terms I-III can be copied except that there is now an additional term, the radiation term $\gamma^{(d-2)AB}\gamma_{AB}'^{(d-2)}$, in the component $\psi_+^{(n)}$. Additionally, some terms which vanished before due to Lemma 6.1 are now non-zero. Taking these small modifications into account we find

$$Q^{(2d-4)} = \frac{1}{\sqrt{2}}\left[-\frac{1}{2}\gamma'^{(2d-4)} - (d-2)\alpha^{(2d-6)} + \frac{1}{2}\gamma^{(d-2)AB}\gamma_{AB}'^{(d-2)}\right]|\epsilon|^2 + \mathcal{D}^A\omega_A\,. \tag{7.121}$$

Substituting (6.20), i.e.,

$$\gamma^{(2d-4)} = \frac{3d-10}{8(d-3)}\gamma^{AB(d-2)}\gamma_{AB}^{(d-2)}\,,$$

, this is equal to

$$Q^{(2d-4)} = \frac{1}{\sqrt{2}} \left[\frac{d-2}{8(d-3)} \gamma^{(d-2)}_{AB} \gamma'^{(d-2)AB} - (d-2)\alpha^{(2d-6)} \right] |\epsilon|^2 .$$

Choosing the normalization $|\epsilon|^2 = \sqrt{2}/8\pi$ we therefore have

$$Q^{(2d-4)} = \frac{1}{8\pi} \left[\frac{d-2}{8(d-3)} \gamma^{(d-2)}_{AB} \gamma'^{(d-2)AB} - (d-2)\alpha^{(2d-6)} \right] . \tag{7.122}$$

Now, with (7.120) and (7.122) at hand, we can use this result to evaluate the integral (7.46). We immediately see that the integral does indeed exist since

$$\lim_{r\to\infty} \left[r^{d-2} \int_\Sigma Q(\tilde{e}^+, \tilde{e}^-) dS_{\tilde{g}} \right] = \int_\Sigma Q^{(2d-4)} \sqrt{s} d^{d-2}x . \tag{7.123}$$

Here it was crucial that terms $Q^{(n)}$ with $n < 2d-4$ are total derivatives since this are the terms which would be divergent in the limit $r \to \infty$. But, by being total derivatives they, luckily, vanish under the integral and (7.123) exists. Recall that we already showed in section 7.3 that

$$\lim_{r\to\infty} \left[r^{d-2} \int_\Sigma Q(\tilde{e}^+, \tilde{e}^-) dS_{\tilde{g}} \right] \geq 0 . \tag{7.124}$$

If we compare the expression for the Bondi mass m_Σ of Σ in (6.45) and (7.122) we see that

$$\int_\Sigma Q^{(d-2)} \sqrt{s} d^{d-2}x = m_\Sigma . \tag{7.125}$$

Therefore, we find

$$m_\Sigma \geq 0 , \tag{7.126}$$

the Bondi mass cannot become negative in odd dimensions $d \geq 5$. This concludes the proof and establishes the main result of this thesis. □

In this chapter, we proofed that, if there exists a Witten spinor near infinity, the Bondi mass is positive in odd dimensions. We discuss the result in the following chapter.

CHAPTER 8

Discussion

This chapter contains a discussion of our main results and we compare them to the results of other authors. Since the results in the chapters 6 and 7 are relatively independent we discuss them separately.

8.1 Discussion of Chapter 6

The asymptotic expansion of the metric components (5.14) we chose are similar to the ones used by Tanabe et al. [105]. However, more assumptions were made in [105]. For example, it was assumed by them that (in our notation)

$$\gamma_{AB}^{(n)} = 0 \quad 1 \leq n \leq d-3\,, \tag{8.1}$$

whereas we did not assume this but derived it in Lemma 6.1 from the vacuum Einstein equations. The assumptions we made were basically all necessary for a Killing spinor to exist near infinity which is crucial for our positivity proof and we do not think that they are very restrictive. In particular, they include a spacetime which is asymptotically Minkowski as a special case. Our computations and results are analogues to the ones by Hollands and Thorne [19, 110] who considered the case of even dimensions. In particular, the Bondi mass takes the form (6.45) in both cases. Our expression for the Bondi mass and the mass-loss formula is also found by [105], albeit by a different method with more assumptions and less explicit, see also [106]. This and the reasonable physical interpretation indicate that, although (6.41) was derived assuming even dimensions, it is possible to use the result in odd dimensions, too. Since the same final result, (6.45), was found in even dimensions in [19] it can be taken as a satisfying expression for the Bondi mass in all dimensions $d \geq 4$. Setting $N_{AB} = 0$ one arrives at the ADM mass. Looking at the results the reason for the difficulties which appear in higher dimension but not in four dimensions becomes apparent. The physically relevant components of the asymptotic expansion were found to be the Coulomb term $\alpha^{(2d-6)}$, at order r^{3-d}, and $\gamma_{AB}^{(d-2)}$, at order $r^{1-d/2}$, which appears in the radiational term. However, for $d = 4$ both terms are of the same order and they are terms of sub-leading order in the asymptotic expansion. Thus, the terms which were problematic by contributing potentially divergent terms to the integral defining the Bondi mass are non-existent in four dimensions. Additionally, the results in Lemma (6.1) are largely trivial in this case. In contrast, in higher dimensions there are potentially divergent terms which, as we showed in chapter 6, vanish, but this result is non-trivial since the physically relevant terms are a priori "hidden" deep inside the asymptotic expansion and only the results of Lemma 6.1 show that terms of higher order are vanishing or "irrelevant" (under the integral). However, in the end, the final results may be viewed as saying that the physics is the same in all dimensions meaning that a Coulomb term and a radiation term contribute to the Bondi mass and these terms are the "relevant sub-leading order terms" (under the integral) in the respective asymptotic expansion.

8.2 Discussion of Chapter 7

To show positivity of the Bondi mass we made two crucial assumptions. First, that there exists a $(d-2)$-dimensional, i.e., odd dimensional, spin manifold Σ admitting a (real) Killing spinor ϵ. Second, that near infinity a Witten spinor exists. As mentioned, assuming that there exists a Killing spinor on Σ implies some restrictions on the possible geometries of Σ. The reason and theory behind this statement is briefly discussed in appendix (D). The result is that, if ϵ is not a parallel spinor, Σ can only be the standard sphere if $d \neq 4m+1, 4m+3$, where $m \geq 1$ is an integer [113]. In this case the manifold is Minkowski near infinity. If $d = 4m+1, 4m+3$ there are many more possibilities for the geometry of Σ. We only give a few examples, see appendix (D) and the references mentioned there for more examples and further discussion. If $d = 7$, Σ can be $S^2 \times S^3$. For $d = 9$, it can, for example, be $SO(5)/SO(3)$, $Sp(2)/Sp(1)$ or the Aloff-Wallach manifolds $N_{k,l} = SU(3)/S^1$, $(k,l) \neq (1,1)$ integers, where the inclusion $S^1 \to SU(3)$ is given by

$$z \mapsto \begin{pmatrix} z^k & 0 & 0 \\ 0 & z^l & 0 \\ 0 & 0 & z^{-l-k} \end{pmatrix}$$

[113]. Examples of possible geometries if Σ admits a parallel Killing spinor are given in [114], see also appendix (D).

A Witten spinor exists near infinity if there exists a solution of (7.36) there. Note that we did not proof that a solution of this elliptic differential equation exits. Thus, our proof of the positivity rests on the assumption that such a spinor exists but a proof of the existence remains to be found.

The difficulties of the positivity proof in higher dimensions are the same we discussed in the previous section. Namely, there are coefficients in the asymptotic expansion of the Witten spinor which are potentially divergent. As discussed in section 7.3, showing the positivity of the Witten-Nester 2-form is rather straightforward in four dimensions and not substantially altered in higher dimensions. The difference between the dimensions only occurs in the step showing that the integral over Q is asymptotically equal to the Bondi mass. Thus, the main part of chapter 7 was to establish Lemma (7.3), which was then used to show that the a priori divergent terms in Q all vanish in the limit $r \to \infty$. In four dimensions these difficulties do not occur and establishing that the Witten-Nester 2-form is asymptotically equal to the Bondi mass is a result obtained relatively easily, but in higher dimensions this result is obtained only after lengthy calculations. The modifications of the Witten's proof [14] in four dimensions, see section 4.3, necessary in higher dimensions due to the complications mentioned above were first made by Hollands and Thorne [19] who proofed that the Bondi mass is positive in higher even dimensions. Since they used the framework of conformal null infinity the result did not hold for odd dimensions [18]. We adopted their arguments to odd dimensions essentially showing that the results are not different in even and odd dimensions which is a non-trivial result since it is not clear at all that this is always the case as can, for example, be seen from the existence of smooth null infinity in all even dimensions but not in odd dimensions [18], or the discussion of black holes in higher dimensions [108]. However, in the present case the results are very similar in both cases. A crucial difference might be establishing the existence of a Witten spinor since the proof in [19] does not carry over to odd dimensions.

CHAPTER 9

Summary and Outlook

In chapter 3 we motivated and introduced the concept of Clifford algebras and some related notions, most importantly how to construct a spinor field on an arbitrary spin manifold, in particular on curved spacetimes, and the definition of a Killing spinor. In section 4.1, Bondi coordinates were introduced and used to define asymptotically flat spacetimes. It was discussed why using the conformal framework is not desirable in odd dimensions. Section 4.2 contains a discussion of why defining mass in general relativity is an issue and reviews the most important definitions of mass in four dimensions, with the focus being on the Bondi mass. A brief discussion of the numerous paper which established that the Bondi mass is positive in four dimensions in section 4.3 concluded the first part. In the second part we derived a coordinate expression for the Bondi mass and established its positivity in odd dimensions $d \geq 5$. Our assumptions and general setup is summarized in chapter 5. Section 5.1 contains a brief discussion of gravitational waves in odd dimensions and shows that the leading order component of the linear perturbation is of half-integer order in r. This is as a motivation for our ansatz for the asymptotic expansion of the metric coefficients in Bondi coordinates. This ansatz was used in section 6.1 to investigate closer Bondi coordinates in odd dimensions in spacetimes where the vacuum Einstein equations hold. With this, a coordinate expression of the Bondi mass was found in section 6.2. Finally, in chapter 7 we showed that the Bondi mass is positive in odd dimensions under some rather loose assumptions. This establishes the most important result of this thesis which was discussed in chapter 8.

The most crucial open issue concerns the Witten spinor. We think that a rigoros proof of the existence of such a spinor ought to be possible but it was not investigated in this work. This result is necessary to complete the proof of positivity. Further investigation and justification of the asymptotic expansion (7.47) of the Witten spinor might also be interesting, for example, a consistency check in the manner described in chapter 6. Furthermore, considering the case of angular momentum, which is closely related to the mass, more closely might be insightful, some results were already obtained by [105]. Additionally, an analysis of the Hamiltonian framework and the derivation of a geometric expression for the Bondi mass in odd dimensions is potentially interesting. Finally, the relation of our results to supergravity was not considered in much detail. As mentioned, the original motivation for Witten's positivity proof came from supergravity and the existence of Killing spinors and the geometry of Σ is closely related to this topic. A discussion of this topic might be nice to see.

Part III

Appendix

Components of Riemann Tensor

Here, we collect the components of the Ricci tensor, adapted from [37]. Indices in this chapter are raised/lowered by γ_{AB}, D_A is the covariant derivative of γ_{AB}.

$$\begin{aligned} R_{rA} &= r\frac{d-2}{2}\beta_A + r^2\left[\frac{1}{4}\beta_A\gamma^{BC}\dot{\gamma}_{BC} - \frac{d-2}{2}\dot{\beta}_A + \frac{d-1}{2}\beta^A\dot{\gamma}_{AB} - \gamma^{BC}D_{[A}\dot{\gamma}_{|C|B]}\right] \\ &+ r^3\left[-\frac{1}{2}\ddot{\beta}_A + \frac{1}{2}\partial_r\left(\gamma^{BC}\beta_C\gamma_{\dot{A}B}\right) - \frac{1}{4}\gamma^{CD}\dot{\gamma}_{CD}\dot{\beta}_A + \frac{1}{4}\gamma^{CD}\dot{\gamma}_{CD}\beta^B\dot{\gamma}_{AB}\right] \end{aligned} \tag{A.1}$$

$$R_{rr} = -r^3\gamma^{AB}\dot{\gamma}_{AB} - \frac{r^4}{4}\left[2\gamma^{AB}\ddot{\gamma}_{AB} - \gamma^{CA}\gamma^{DB}\dot{\gamma}_{AB}\dot{\gamma}_{CD}\right] \tag{A.2}$$

$$\begin{aligned} R_{AB} = &-\frac{1}{r}\beta^C\beta_C\gamma_{AB} + \mathcal{R}_{AB} - \frac{1}{2}\beta_A\beta_B + (d-1)D_{(B}\beta_{A)} - 2(d-3)\alpha\gamma_{AB} \\ &- \gamma_{AB}\beta^C\beta_C - \frac{1}{2}\beta^C D^D\gamma_{CD}\gamma_{AB} + \frac{1}{2}\gamma^{CD}D_{(E}\beta_{C)}\gamma_{AB} \\ &+ r\left[\frac{d-2}{2}\gamma'_{AB} + \frac{1}{2}\gamma_{AB}\gamma^{CD}\gamma'_{CD} + \frac{1}{2}D_C\left(\gamma^{CD}\beta_D\dot{\gamma}_{AB}\right) + \frac{1}{2}\gamma^{CD}\dot{\gamma}_{CD}D_{(A}\beta_{B)}\right. \\ &+\gamma^{EF}\beta_E\beta_F\dot{\gamma}_{AB} + \partial_r(D_{(A}\beta_{B)}) + \beta_{(A}\dot{\beta}_{B)} - \beta^C\beta_{(A}\dot{\gamma}_{B)C} - (D^C\beta_{(A})\dot{\gamma}_{B)C} - (d-2)\alpha\dot{\gamma}_{AB} \\ &-\frac{d-2}{2}\beta^C\beta_C\dot{\gamma}_{AB} - \gamma_{AB}\alpha\gamma^{CD}\dot{\gamma}_{CD} - \frac{1}{2}\gamma_{AB}\beta^C\beta_C\dot{\gamma}_{DE}\dot{\gamma}^{DE} - 2\gamma_{AB}\dot{\alpha} + \gamma_{AB}\beta^C\beta^D\dot{\gamma}_{CD} \\ &\left.-2\beta^E\beta_E\dot{\gamma}_{AB}\right] + r^2\left[\dot{\gamma}'_{AB} - \gamma^{CD}\dot{\gamma}_{D(A}\gamma'_{B)C} + \frac{1}{4}\gamma^{CD}\gamma'_{CD}\dot{\gamma}_{AB} + \frac{1}{4}\gamma^{CD}\gamma'_{AB}\dot{\gamma}_{CD}\right. \\ &-\alpha\ddot{\gamma}_{AB} - \frac{1}{4}\gamma^{CD}\dot{\gamma}_{CD}\beta^E\beta_E\dot{\gamma}_{AB} - \frac{1}{2}\gamma^{CD}\dot{\gamma}_{CD}\alpha\dot{\gamma}_{AB} - \dot{\alpha}\dot{\gamma}_{AB} - \frac{1}{2}\partial_r(\gamma^{EF}\beta_E\beta_F)\dot{\gamma}_{AB} \\ &-\frac{1}{2}\beta^E\beta_E\ddot{\gamma}_{AB} - \frac{1}{2}\dot{\beta}_A\dot{\beta}_B - \frac{1}{2}\beta^E\beta^F\dot{\gamma}_{AE}\dot{\gamma}_{BF} + \beta^C\dot{\beta}_{(A}\dot{\gamma}_{B)C} + \frac{1}{2}\gamma^{CD}\beta^E\beta_E\dot{\gamma}_{CA}\dot{\gamma}_{DB} \\ &\left.+\alpha\gamma^{CD}\dot{\gamma}_{CA}\dot{\gamma}_{DB}\right] \end{aligned} \tag{A.3}$$

$$\begin{aligned} R_{ur} = &-\frac{1}{r}\frac{d-2}{2}\beta^A\beta_A + \frac{d-2}{2}\beta^A\beta_A + \frac{1}{2}\gamma^{CE}D_{(E}\beta_{C)} - \gamma^{AB}\gamma_{AB}\beta^C\beta_C \\ &+ r\left[\gamma^{AB}\gamma'_{AB} + \gamma^{AB}\dot{\gamma}_{AB}\alpha + \frac{1}{4}\gamma^{AB}\dot{\gamma}_{AB}\beta^C\beta_C + 2\beta^A\dot{\beta}_A + \frac{1}{2}\dot{\gamma}^{AB}\beta_A\beta_B + \frac{1}{2}D^A\dot{\beta}_A - (d-2)\dot{\alpha}\right. \\ &\left.-\alpha\gamma^{AB}\dot{\gamma}_{AB} - \frac{1}{2}\beta^C\beta_C\dot{\gamma}_{AB}\gamma^{AB} + \frac{1}{2}\beta^A\beta^B\dot{\gamma}_{AB}\right] \\ &+ r^2\left[-\frac{1}{4}\gamma^{CA}\gamma^{DB}\gamma'_{CD}\dot{\gamma}_{AB} + \frac{1}{2}\gamma^{AB}\dot{\gamma}'_{AB} - \ddot{\alpha} - \frac{1}{2}\gamma^{AB}\dot{\gamma}_{AB}\dot{\alpha} - \frac{1}{4}\gamma^{AB}\dot{\gamma}_{AB}\beta^C\dot{\beta}_C - \frac{1}{2}\partial_r(\beta^A\dot{\beta}_A)\right] \end{aligned} \tag{A.4}$$

$$
\begin{aligned}
R_{uA} = \frac{1}{r}\Big[&\frac{1}{2}\beta_A\beta^C\beta_C - 2\beta^B D_{[A}\beta_{B]} + \frac{1}{2}\gamma^{BC}D_B(\beta_C\beta_A) - D^C D_{[A}\beta_{C]} - \frac{1}{2}\beta^C D_C\beta_A \\
&+(d-3)D_A\alpha - \beta_A\alpha - \frac{1}{2}\beta^C\beta_C\beta_A - \beta_A\gamma^{CE}D_{(E}\beta_{C)} + 2\alpha\beta_A\gamma^{CD}\gamma_{CD} + \gamma^{CD}\gamma_{CD}\beta^B\beta_B\beta_A \\
&+2(d-2)\beta_A + (d-2)\beta_A\beta^C\beta_C\Big] - D_A\alpha + \frac{1}{2}\beta'_A + \frac{1}{4}\beta_A\gamma^{BC}\gamma'_{BC} - (D_{[A}\gamma'_{|C|B]})\gamma^{BC} \\
&- \frac{1}{2}\beta_A\partial_r(\gamma^{BC}\beta_B\beta_C) - \frac{1}{2}\beta^C\beta_C\dot\beta_A - \frac{1}{2}\gamma^{CD}\gamma'_{CD}\beta_A + D_A\dot\alpha + \partial_r(\beta^D D_{[A}\beta_{D]}) - \frac{1}{4}\beta_A\gamma^{BC}\dot\gamma_{BC}\beta^E\beta_E \\
&- \frac{1}{2}\alpha\gamma^{BC}\dot\gamma_{BC} + \frac{1}{2}\gamma^{BC}\dot\gamma_{BC}D_A\alpha + \frac{1}{2}\gamma^{BC}\dot\gamma_{BC}\beta^E D_{[A}\beta_{E]} - \frac{1}{2}D_B(\gamma^{BC}\beta_C\dot\beta_A) + \frac{1}{2}\beta^C\dot\gamma_{CA}\beta^E\beta_E \\
&+ \beta^C\alpha\dot\gamma_{CA} - \frac{1}{2}\beta^C\beta_C\beta^E\dot\gamma_{CA} + \frac{1}{2}\dot\beta^B D_C\beta_A + \frac{1}{2}\dot\beta^B\beta_B\beta_A - 2\alpha\beta^C\dot\gamma_{CA} - \dot\gamma_{CA}D^C\alpha \\
&- \dot\gamma_{CA}\beta^D\gamma^{BC}D_{[B}\beta_{D]} + \alpha\beta_A\gamma^{CD}\dot\gamma_{CB} + \frac{1}{2}\beta^C\beta_C\dot\gamma_{DE}\gamma^{DE}\beta_A + \dot\alpha\beta_A - \frac{1}{2}\beta_A\beta^B\beta^C\dot\gamma_{BC} + \beta_A\beta^C\dot\beta_C \\
&+ \frac{d-2}{2}\beta^C\beta_C\dot\beta_A + r\Big[-\frac{1}{4}\gamma^{BC}\gamma'_{BC}\dot\beta_A - \frac{1}{2}\dot\beta'_A - \frac{1}{2}\partial_r(\gamma^{BC}\beta_B\gamma'_{CA}) - \frac{1}{4}\gamma^{BC}\dot\gamma_{BC}\beta^E\gamma'_{EA} + \frac{1}{2}\dot\beta^C\gamma'_{CA} \\
&+\frac{1}{2}\gamma^{BC}\dot\gamma_{CA}\beta'_B - 2(d-2)\alpha\dot\beta_A + \frac{1}{2}\dot\beta_A\partial_r(\gamma^{BC}\beta_B\beta_C) + \frac{1}{4}\gamma^{BC}\dot\gamma_{BC}\beta^E\beta_E\dot\beta_A + \frac{1}{2}\gamma^{BC}\dot\gamma_{BC}\alpha\dot\beta_A \\
&-\frac{1}{2}\dot\beta^B\beta^E\beta_E\dot\gamma_{CA} - \dot\beta_C\alpha\dot\gamma_{CA} - \frac{1}{2}\dot\beta^C\beta_C\dot\beta_A + \frac{1}{2}\dot\beta^C\beta_C\beta^E\dot\gamma_{EA} + \beta^C\dot\gamma_{CA}\dot\alpha\Big] - r^2(d-2)\alpha\ddot\beta_A
\end{aligned}
\tag{A.5}
$$

$$
\begin{aligned}
R_{uu} = \frac{1}{r^2}\Big[&2\alpha D^A\beta_A + 3\beta^A D_A\alpha + D^A D_A\alpha + 11\alpha\beta^A\beta_A + 2\beta^A\beta^B D_{[A}\beta_{B]} - \dot\gamma^{AB}\beta_A\beta'_B - (d-2)\beta^A D_A\alpha \\
&-2\alpha\beta^A\beta_A - 2\alpha\gamma^{CE}D_{(E}\beta_{C)} + 4\alpha^2\gamma^{AB}\gamma_{AB} + 2\gamma^{AB}\gamma_{AB}\beta^C\beta_C - 4(d-2)\alpha^2 - (d-2)2\alpha\beta^A\beta_A\Big] \\
&+ \frac{1}{r}\Big[-D^A\beta'_A - \beta^A\beta'_A + (d-2)\alpha' + (d-2)\beta^A\beta'_A - 4\alpha\dot\alpha - \frac{1}{2}\beta^D(D_D\alpha)\gamma^{AB}\dot\gamma_{AB} - D^A(\beta_A\dot\alpha) \\
&-\frac{1}{2}\beta^B\dot\beta_B\beta^E\beta_E - 2\alpha\dot\beta_B\beta^B - \frac{1}{2}\dot\beta^B\beta_B\beta^E\beta_E + 2\dot\beta^B D_B\alpha - 2\dot\beta^B\beta^A D_{[A}\beta_{B]} - \alpha\partial_r(\beta^A\beta_A) - 4\dot\alpha\beta^A\beta_A \\
&+\beta^A\beta_A\beta^C\dot\beta_C - 2\beta^A\beta_A\dot\alpha - 4\beta^A\dot\beta_A\alpha - \partial_r(\beta^A D_A\alpha) - \partial_r(\beta^A\beta_A) + 2(d-2)\alpha\dot\alpha + 2\alpha^2\gamma^{AB}\dot\gamma_{AB}\Big] \\
&+ \beta^C\beta_C\dot\gamma_{AB}\gamma^{AB}\alpha + 2\alpha\dot\alpha + 2\alpha\beta^A\dot\beta_A - \alpha\beta^A\beta^B\dot\gamma_{AB} - \frac{1}{2}\gamma^{AB}\gamma''_{AB} + \frac{1}{4}\gamma'^{AB}\gamma'_{AB} - \frac{1}{2}\dot\alpha\gamma^{AB}\gamma'_{AB} \\
&+ \frac{1}{2}\gamma^{AB}\dot\gamma_{AB}\alpha' + \frac{1}{2}\beta^C\beta'_C\gamma^{AB}\dot\gamma_{AB} - \dot\alpha\beta^C\beta_C\gamma^{AB}\dot\gamma_{AB} - 2\alpha\dot\alpha\gamma^{AB}\dot\gamma_{AB} - 2\dot\alpha\beta^A\beta_A \\
&+ \beta^A\dot\beta'_A + 2\beta^A\beta_A\dot\alpha + \frac{1}{2}\dot\beta^B\dot\beta_B\beta^E\beta_E + \alpha\dot\beta^B\dot\beta_B + \dot\alpha\partial_r(\beta^A\beta_A) - \frac{1}{2}\beta^A\dot\beta_A\beta^C\dot\beta_C + 2\dot\alpha\beta^A\dot\beta_A \\
&+ r\Big[-4\dot\alpha\beta^A\beta_A + 2\alpha\ddot\alpha + \beta^A\beta_A\ddot\alpha\Big] + r^2\Big[-\beta^C\beta_C\dot\alpha\gamma^{AB}\dot\gamma_{AB} - 2\alpha\dot\alpha\gamma^{AB}\dot\gamma_{AB}\Big] \\
&- r^3\Big[-\frac{1}{2}\beta^C\beta_C\ddot\alpha\gamma^{AB}\dot\gamma_{AB} - \alpha\ddot\alpha\gamma^{AB}\dot\gamma_{AB}\Big]
\end{aligned}
\tag{A.6}
$$

Physical and Unphysical Derivative of Spinor

We want to derive (7.34), i.e., an expression relating

$$\nabla_a \psi = \partial_a \psi + \omega_a \psi \qquad \text{and} \qquad \tilde{\nabla}_a \tilde{\psi} = \tilde{\partial}_a \psi + \tilde{\omega}_a \psi \,. \tag{B.1}$$

If we subtract the second equation from the first this yields

$$(\nabla_a - \tilde{\nabla}_a)\psi = (\omega_a - \tilde{\omega}_a)\psi \tag{B.2}$$

since $\partial_a = \tilde{\partial}_a$. Thus, we would like an explicit expression for

$$\omega_a - \tilde{\omega}_a = \frac{1}{8}\left\{ r^{-1}\tilde{e}^{\mu b}\nabla_a \left(r\tilde{e}^{\nu}_{b}\right) - \tilde{e}^{\mu b}\tilde{\nabla}_a \tilde{e}^{\nu}_{b}\right\}[\sigma_\mu, \sigma_\nu]$$

where $\omega^{\mu\nu}_a = \frac{1}{8}e^{\mu b}\nabla_a e^{\nu}_b$ and $e^{\nu}_b = r\tilde{e}^{\nu}_b$ used. Using $\lambda_{\mu\nu} = e^{a}_{\mu} e_{\nu a}$ this can be rewritten as

$$\omega_a - \tilde{\omega}_a = \frac{1}{8}\left\{ r^{-1}\lambda^{\mu\nu}\nabla_a r + \tilde{e}^{\mu b}(\nabla_a - \tilde{\nabla}_a)\tilde{e}^{\nu}_{b}\right\}[\sigma_\mu, \sigma_\nu]\,.$$

To further simplify this, we use the formula for conformal transformation of the covariant derivative and find

$$\begin{aligned}\nabla_a \tilde{e}^{\nu}_b &= \tilde{\nabla}_a \tilde{e}^{\nu}_b - \frac{1}{2}r^{-2}\tilde{g}^{cd}\left(\tilde{\nabla}_a(r^2\tilde{g}_{bd}) + \tilde{\nabla}_b(r^2\tilde{g}_{ad}) - \tilde{\nabla}_d(r^2\tilde{g}_{ab})\right)\tilde{e}^{\nu}_c \\ &= \tilde{\nabla}_a \tilde{e}^{\nu}_b - r^{-1}\left(\delta^c_b\tilde{\nabla}_a r + \delta^c_a \tilde{\nabla}_b r - \tilde{g}_{ab}\tilde{\nabla}^c r\right)\tilde{e}^{\nu}_c\,.\end{aligned}$$

Hence,

$$\omega_a - \tilde{\omega}_a = \frac{1}{8}\left\{ r^{-1}\lambda^{\mu\nu}\nabla_a r - r^{-1}\tilde{e}^{\mu b}\left(\delta^c_b\tilde{\nabla}_a r + \delta^c_a\tilde{\nabla}_b r - \tilde{g}_{ab}\tilde{\nabla}^c r\right)\tilde{e}^{\nu}_c\right\}[\sigma_\mu, \sigma_\nu]\,. \tag{B.3}$$

The first two terms on the right side cancel and the remaining expression can be written as

$$\begin{aligned}\omega_a - \tilde{\omega}_a &= -\frac{1}{8r}\tilde{e}^{\mu}_b\tilde{e}^{\nu}_c\left\{\delta^c_a\tilde{\nabla}^b r - \tilde{g}^b_a\tilde{\nabla}^c r\right\}[\sigma_\mu, \sigma_\nu] \\ &= \frac{1}{4r}\tilde{e}^{[\mu}_a\tilde{e}^{\nu]}_b\left\{\tilde{\nabla}^b r\right\}[\sigma_\mu, \sigma_\nu] \\ &= -\frac{1}{4r}\tilde{e}^{[\mu}_a\tilde{e}^{\nu]}_b\left\{\tilde{\nabla}^b r\right\}(\{\sigma_\nu, \sigma_\mu\} - 2\sigma_\mu\sigma_\nu) \\ &= -\frac{1}{2r}\left(\tilde{g}_{ab}\tilde{\nabla}^b r - \tilde{\sigma}_a\tilde{\sigma}_b\tilde{\nabla}^b r\right)\end{aligned} \tag{B.4}$$

and therefore (B.2) can be written as

$$\nabla_a\psi = \tilde{\nabla}_a\psi - \frac{1}{2r}\left\{\left(\tilde{\nabla}_a r\right) - \tilde{\sigma}_a\tilde{\sigma}_b\left(\tilde{\nabla}^b r\right)\right\}\psi\,. \tag{B.5}$$

Substituting

$$\tilde{\nabla}_a \psi = \tilde{\nabla}_a (r^{1/2} \tilde{\psi}) = r^{1/2} \tilde{\nabla}_a \tilde{\psi} + \frac{1}{2} r^{-1/2} (\tilde{\nabla}_a r) \tilde{\psi} \tag{B.6}$$

one finds

$$\nabla_a \psi = r^{1/2} \left(\tilde{\nabla}_a \tilde{\psi} + \frac{1}{2r} \tilde{\sigma}_a \tilde{\sigma}_b (\tilde{\nabla}^b r) \right) \tilde{\psi} \, . \tag{B.7}$$

This is equation (7.34).

APPENDIX C

Local Symmetries and Conserved Quantities

In this chapter we sketch the construction behind the Hamiltonian formalism, which is used to derive the geometric expression for the Bondi mass. As a motivation we show how the symplectic structure of classical mechanics leads to the familiar equations that are usually derived without reference to the underlying manifold structure, see e.g. [22, 115, 116]. Then, we review how to derive conserved quantities and, in particular, the Bondi mass in higher dimensions. We skip many details, such as convergence issues, uniqueness etc., and refer to the references [17, 56, 117], which we follow closely in the following.

C.1 Hamiltonian Mechanics and Symplectic Manifolds

As a motivation we consider the symplectic structure present in classical mechanics without much rigor and the Hamiltonian to be defined is time-independent. Let M be a smooth manifold and $T^\star M$ its cotangent bundle called **phase space**. Locally, we have a chart on $T^\star\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$ with coordinates $((x^i), (p_j))$. There is a unique 1-form on $T^\star M$ and in coordinates the 1-form is

$$\theta = p_i dx^i \, . \tag{C.1}$$

It is called canonical 1-form, Poincaré 1-form, Liouville 1-form or **symplectic potential**. The latter name is due to the fact that the symplectic form can be defined as

$$\omega^2 = d\theta = dp_i \wedge dq^i \, . \tag{C.2}$$

This is a symplectic form since $d\omega^2 = 0$ and ω^2 is nondegenerate, that is

$$\forall \xi \neq 0 \; \exists \eta : \omega^2(\xi, \eta) \neq 0 \tag{C.3}$$

where $\xi, \eta \in TM$ (in local coordinates $\det \omega_{ij} \neq 0$) and thus $(T^\star M, \omega^2)$ is a symplectic manifold. Due to this structure, to each vector ξ tangent to the symplectic manifold at point x there is an associated 1-form ω^1_ξ on $T_x M$ given by the formula $\omega^1_\xi(\eta) = \omega^2(\eta, \xi)$ for all $\eta \in T_x M$. Since ω_{ab} is non-degenerate there is an inverse ω^{ab} of ω_{ab}, i.e. $\omega^{ab}\omega_{bc} = \delta^a_c$, and this induces a (fiberwise) isomorphism $I : T^*_x M \to T_x M$. In coordinates, the isomorphism reads $\xi^a = \omega^{ab}(\omega^1_\xi)_b$. That is, if we have a 1-form there is a corresponding vector field. In this sense, the symplectic form acts like a metric. We will now use this to make a connection with the usual Hamiltonian equation of motions. Let $H : M \to \mathbb{R}$ be a function. Then at each point there is a tangent vector

$I(dH)$, associated to dH, which defines a vector field $I(dH) \equiv X_H$ on M where $\omega^2(X_H, -) = \omega^1_{X_H}(-) = dH(-)$ or, equivalently, $(X_H)^a = \omega^{ab}(dH_\xi)_b$. H is called **Hamiltonian** and the associated vector field X_H is called Hamiltonian vector field if there is a 1-parameter group of diffeomorphisms $g_t : M \to M$ induced by X_H such that

$$\left.\frac{d}{dt}\right|_{t=0} g_t \cdot x = X_H(x)\,. \tag{C.4}$$

This preserves the symplectic structure $(g_t)^*\omega^2 = \omega^2$. An equivalent condition is $\mathcal{L}_{X_H}\omega^2 = 0$ and for $d = 1$ this is just Liouville's theorem. g_t is called Hamiltonian flow. To summarize, we have the following maps

$$C(M) \ni H \xrightarrow{\ d\ } (dH)_b \in T^\star M \xrightarrow{\ I\ } (I(dH))^a = (X_H)^a \in TM\,.$$

Now, solutions to the equations of motion are trajectories $\gamma : \mathbb{R} \to M$ which satisfy $(dH)_b = \omega_{ab}\dot{\gamma}^a$. In classical mechanics, where $\gamma(t) = (p(t), q(t))$ is some trajectory in phase space, this yields the well known Hamilton equations

$$\dot{\gamma} = X_H(x) \Leftrightarrow \dot{p} = -\frac{\partial H}{\partial q} \quad \text{and} \quad \dot{q} = \frac{\partial H}{\partial p}\,, \tag{C.5}$$

i.e., γ is an integral curve of the Hamiltonian vector field if and only if it solves the equations on the right side. Using the usual definition $H = \dot{q}^i p_i - L$ one can also derive these equations from computing δH using $p^i = \partial L/\partial \dot{q}_i$ and $\dot{p}^i = \partial L/\partial q_i$, thus $\delta H = \dot{q}^i \delta p_i - \dot{p}^i \delta q_i$ which gives the same equations.

The Poisson bracket $\{-, -\}$ of functions F and H is

$$\{F, H\} = \omega^2(X_H, X_F) = \mathcal{L}_{X_F} H \tag{C.6}$$

and if $\{F, H\} = 0$ then F is constant along the integral curves of H and vice versa (essentially Noether's theorem).

C.2 Covariant Phase Space

In this section we define some general concepts generalizing the definitions in C.1 to theories other than classical mechanics. In particular, the definitions are applicable to general relativity. We follow [117] closely. There are not many references to general relativity in this section and we will discuss the application of this general construction and its use in general relativity in the subsequent sections.

We consider a d-dimensional spacetime M with topology $\mathbb{R} \times \mathcal{T}$ where M is a globally hyperbolic d-dimensional spacetime and each slice $\mathcal{T}_t$ of the foliation of $\mathbb{R} \times \mathcal{T}$ is a $(d-1)$-dimensional compact submanifold without boundary. $\phi : M \to M$ is a field and the manifold $\mathscr{F}$

is the set of all field configurations, i.e., ϕ is represented by a point in $\mathscr{F}$. In general relativity, the field variable is the spacetime metric. There is an action S which is a functional on $\mathscr{F}$, i.e. $S : \mathscr{F} \to \mathbb{R}$, and it is defined by $S[\phi] = \int_M \boldsymbol{L}$ where $\boldsymbol{L}$ is the n-form Lagrangian density. The boldface implies that there is an implicit volume form, i.e. $\boldsymbol{L} = L\epsilon_{a_1\ldots a_d}$ with L the scalar Lagrangian density. Thus, $\boldsymbol{L}$ is an example of a scalar density, which can be integrated over M. A vector density $\boldsymbol{v}^b$ is defined similarly, $\boldsymbol{v}^b = v^b\epsilon_{a_1\ldots a_d}$ and this can be made into an $(d-1)-$form by contracting with the first index, i.e. $\boldsymbol{v} = v^b\epsilon_{ba_2\ldots a_d}$. Thus, there is no notational difference between $d-$forms, $(d-1)-$forms etc., but it should be clear from the context what is meant by a given boldface letter.

Let there be a smooth one-parameter family $\phi(\lambda) : M \to M$. The first variation of the Lagrangian density about the field configuration $\phi_0 = \phi(0)$ is

$$\delta\boldsymbol{L} \equiv \left.\frac{d}{d\lambda}\boldsymbol{L}\right|_{\lambda=0} = \boldsymbol{E}(\phi)\delta\phi + d\boldsymbol{\theta}(\phi,\delta\phi) \tag{C.7}$$

Here, $\delta\phi^a(x)$ is a tangent to the curve $c(\lambda) = \phi(\lambda, x)$ (with x fixed) in M at the point $\lambda = 0$. Thus, $\delta\phi^a(x)$ may be viewed as a vector in the tangent space to M at the point $\phi_0(x)$. $\boldsymbol{\theta}^\mu$ is called **symplectic potential current density** (which will be justified later). Since the action at field configuration ϕ_0 is stationary ($dS/d\lambda = 0$) for all variations $\delta\phi^a$ if and only if $\boldsymbol{E}_a\delta\phi^a = 0$ at ϕ_0, the equation $\boldsymbol{E} = 0$ is the **equation of motion** for ϕ. We saw that the field variation $\delta\phi^a$ may be viewed as a vector in the tangent space to M at the point $\phi_0(x)$. A different point of view is possible and often preferable. This perspective is to look at the variation $\delta\phi^a$ as a vector in the tangent space to $\mathscr{F}$ at the point ϕ. Using an abstract index notation with capital roman letters for tensor fields on $\mathscr{F}$, we can write $(\delta\phi)^A$ when we view field variations in this manner.

Now, take the equation (C.7) as defining for $\boldsymbol{\theta}$ and define the functional

$$\theta[\phi^a, \delta\phi^a] \equiv \int_{\mathcal{T}} \boldsymbol{\theta}\,, \tag{C.8}$$

where the orientation is chosen to be $n^{\alpha_1}\epsilon_{\alpha_1\ldots\alpha_d}$ with n^a future-directed and timelike. The so defined θ depends on $\mathcal{T}$ and is called **presymplectic potential** . Assuming that θ is continuous it defines a dual vector field on $\mathscr{F}$, denoted θ_A, and given by $\theta_A(\delta\phi)^A = \theta[\phi^a, \delta\phi^a]$ for all $(\delta\phi)^A$. Let d denote the exterior derivative on $\mathscr{F}$ (it should always be clear from context if d is the dimension or the derivative). Then there is a 2-form $\omega_{AB} = \omega_{[AB]}$ on $\mathscr{F}$ defined by

$$\omega_{AB} = (d\theta)_{AB}\,. \tag{C.9}$$

(On the level of densities this is $\boldsymbol{\omega}^a = \delta_1\boldsymbol{\theta}_2^a - \delta_2\boldsymbol{\theta}_1^a$.) By definition it is exact (and in particular closed) and called **presymplectic form** because it has all properties of a symplectic form except that it is degenerate (any $\delta\phi^a$ with support away from $\mathcal{T}$ gives rise to degeneracy direction $(\delta\phi)^A$ for ω_{AB}). This equation also justifies the name "presymplectic potential" for θ. Thus, $(\mathscr{F}, \omega)$ is a presymplectic space. There is a so-called reduction procedure, which is used to create a symplectic space from the presymplectic space. The idea is to "divide out" the degeneracies by defining an equivalence relation on $\mathscr{F}$ by $\phi_1 \simeq \phi_2$ if and only if ϕ_1 and ϕ_2 lie on the same integral submanifold of degeneracy vectors. A integral submanifold is a higher-dimensional version of integral curves. Like integral curves, integral submanifolds are also generated by/associated to a vector field and foliate M. In the present case one looks at the integral submanifold corresponding to the degeneracy vectors; see [117] for the precise argument.

Let Γ denote set of equivalence classes of $\mathscr{F}$ and $\pi : \mathscr{F} \to \Gamma$ the map, which maps each field onto its equivalence class. Assuming that Γ has a manifold structure, it results in $\mathscr{F}$ having the structure of a fiber bundle over Γ with projection π and the fibers being all the fields in an equivalence class. Note that the procedure described is very similar to the construction of principle G–bundles. We use the same index notation for tensors on Γ as for tensors on $\mathscr{F}$. Lastly, define the 2-form Ω_{AB} on Γ by

$$\omega_{AB} = \pi^\star \Omega_{AB} \, . \tag{C.10}$$

Ω_{AB} is closed (by construction) but does not need to be exact. Since we divided out the degeneracies in this construction, we now have the **symplectic manifold** (Γ, Ω_{AB}). Note that it depends on $\mathcal{T}$, as the definition of ω does. We have the following chain of relations

$$\boldsymbol{\theta} \xrightarrow{\int_{\mathcal{T}}} \theta \xrightarrow{d} \omega \xrightarrow{\pi} \Omega \, .$$

So far we have worked only with the second term in the variation of the Lagrange density (C.7). Recall that the first term corresponds to the equation of motion of the field ϕ, i.e., this is the part where the dynamics (and physics) is "hidden". Thus, to describe a physical system we have to also make use of this equation. This leads to the physical phase space, the route being as follows. Let $\bar{\mathscr{F}}$ be the submanifold of $\mathscr{F}$ which consists of all solutions of the equation of motion $\boldsymbol{E}_a = 0$. By restricting the above reduction procedure to $\bar{\mathscr{F}}$ we define $\bar{\Gamma} = \pi[\bar{\mathscr{F}}]$ which is a submanifold of Γ called **constraint submanifold**. Defining a symplectic form on $\bar{\Gamma}$ in the obvious way, namely $\bar{\omega}_{AB} = \bar{\pi}^\star \bar{\Omega}_{AB}$, we have the physical symplectic manifold $(\bar{\Gamma}, \bar{\Omega}_{AB})$ which, in many physical theories, is equal to the phase space. In particular, it can be shown that the definition of $(\bar{\Gamma}, \bar{\Omega}_{AB})$ is independent of $\mathcal{T}$. One may view Γ as the set of all kinematically possible states while $\bar{\Gamma}$ consists of all dynamically possible states. That is, if both are not equal, i.e. if $\bar{\Gamma}$ is a true subset of Γ, then there are (physical) constrains.

So far we only discussed Lagrangians and, to make a connection with classical mechanics and conserved quantities (an example being the Bondi mass), we would like to define a Hamiltonian.

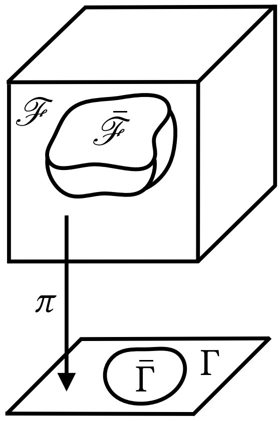


Figure C.1: Projection from space of all states $\mathscr{F}$ and space of all solutions $\bar{\mathscr{F}}$ onto phase space Γ and constrained phase space $\bar{\Gamma}$, respectively.

It is constructed as follows. Let t^a be a complete vector field on M such that the diffeomorphisms $\Lambda(t)$ generated by t^a map Cauchy surfaces into Cauchy surfaces. t^a is called time translation, although it does not have to be timelike. There is a 1-parameter family of field configurations $\phi(t) = \phi \circ \Lambda(t)$ ("time evolution") and the associated variation is denoted $\delta_t \phi^a$. Assume that for each solution $\phi \in \bar{\mathscr{F}}$ we have $(\delta_t \phi)^A \equiv \tau^A$ such that τ^A is a tangent vector field on $\bar{\mathscr{F}}$. This vector field represents time evolution on $\bar{\mathscr{F}}$ induced by t^a and $\mathcal{L}_\tau \bar{\omega}_{AB} = 0$, since $\phi \in \bar{\mathscr{F}}$ and thus ω is independent of $\mathcal{T}$. Now, assume that τ^A is such that it has a well-defined projection to a vector field T^A on $\bar{\Gamma}$. If this is possible and T^A exists it represents time evolution on $\bar{\Gamma}$ and $\mathcal{L}_T \bar{\Omega}_{AB} = 0$. Then, there exists a function H on Γ such that, evaluated on $\bar{\Gamma}$, we have

$$(dH)_A = \Omega_{AB} T^B \leftrightarrow T^A = \Omega^{AB} (dH)_B \,, \tag{C.11}$$

which are Hamilton's equations of motion. Such a Hamiltonian exists only if the projection of the time translation vector field to $\bar{\Gamma}$ is possible, see [117] for details.

C.3 Symmetries at Infinity

We now follow mostly [56]. Our goal is to define the Bondi mass. We have seen in the last two sections that a Hamiltonian, which is related to a conserved quantity, can be associated to a vector field. In the following sections we will show how this can be used to define the Bondi mass, but first we have to describe more precisely the vector fields to be used. Let $\mathcal{B}$ be a boundary of M such that $M \cup \mathcal{B}$ is a d-dimensional manifold with boundary. We consider slices $\mathcal{T}$ in the physical spacetime, M, which extend smoothly to $\mathcal{B}$ in the unphysical spacetime,

$M \cup \mathcal{B}$, such that the extended hypersurface intersects $\mathcal{B}$ smooth in a $(d-2)$-dimensional submanifold, denoted $\partial\mathcal{T}$ and called cross section of $\mathcal{B}$. $\mathcal{T} \cup \partial\mathcal{T}$ is assumed to be compact. An infinitesimal asymptotic symmetry ξ^a is a complete vector field on $M \cup \mathcal{B}$ such that ξ^a is tangent to $\mathcal{B}$ on $\mathcal{B}$. ξ^a is a representation of an infinitesimal asymptotic symmetry if its associated one-parameter group of diffeomorphisms maps $\bar{\mathscr{F}}$ into itself. Equivalently, ξ^a is a representation, if the field variation $\mathcal{L}_\xi\phi$ with $\phi \in \bar{\mathscr{F}}$ is a vector tangent to $\bar{\mathscr{F}}$. In this way, the vector field gives rise to the variations of the field ϕ, which is the connection to the previous section. Two representations ξ^a and ξ'^a are equivalent if they coincide on $\mathcal{B}$ and give rise to the "same transformation" (see [56] for details). The equivalence classes of representatives of infinitesimal asymptotic symmetries are the infinitesimal asymptotic symmetries of the theory. $\delta_\xi\phi = \mathcal{L}_\xi\phi$ may be viewed as the dynamical evolution vector field corresponding to the notion of "translations" generated by ξ^a. Define the **Noether current** $(d-1)$-form (a vector density) associated with ξ^a by

$$\boldsymbol{j} = \boldsymbol{\theta}(\phi, \mathcal{L}_\xi\phi) - \xi \cdot \boldsymbol{L} = d\boldsymbol{Q} \tag{C.12}$$

Here, $\xi \cdot \boldsymbol{L}$ denotes the contraction of ξ with $\boldsymbol{L}$ and we have the **Noether charge**

$$Q = \int_{\mathcal{T}} \boldsymbol{j}^\mu n_\mu \tag{C.13}$$

that is independent of $\mathcal{T}$ only if $\phi \in \bar{\mathscr{F}}$, since $\boldsymbol{j}$ is conserved in this case. If the time evolution vector field induced by ξ^a on $\bar{\Gamma}$ preserves the symplectic form $\bar{\Omega}_{AB}$ the time evolution will be generated by a Hamiltonian H_ξ. In this way we can introduce the notion of a Hamiltonian H_ξ conjugate to the vector field ξ^a (note the similarity to classical mechanics in section C.1).

C.4 Hamiltonian associated with Symmetry

We now make the notion of a Hamiltonian $H_\xi : \bar{\mathscr{F}} \to \mathbb{R}$ associated to the vector field ξ^a at time $\mathcal{T}$ more precise. However, as shown by [56], a Hamiltonian H_ξ corresponding to a conserved quantity exists in general only if the extension of $\boldsymbol{\omega}$ to $\mathcal{B}$ has vanishing pullback to $\mathcal{B}$. This is the case at spatial infinity but not at null infinity. Therefore, a definition of ADM mass is easier than one of Bondi mass (especially in higher dimensions). We consider in the following only the case of null infinity and show how an analog of H_ξ can be defined. We use the notation $\mathcal{H}_\xi$ for the "Hamiltonian-like" function to distinguish it from the true Hamiltonian H_ξ.

On $\mathcal{B}$ let Θ be the symplectic potential for the (at null infinity non-vanishing) pullback $\bar{\boldsymbol{\omega}}$ of the extension of the symplectic current form $\boldsymbol{\omega}$ to $\mathcal{B}$. We require Θ to be independent of the the conformal factor. On $\mathcal{B}$ we have

$$\bar{\boldsymbol{\omega}}(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\Theta(\phi, \delta_2\phi) - \delta_2\Theta(\phi, \delta_1\phi)\,, \tag{C.14}$$

for all $\phi \in \bar{\mathscr{F}}$ and $\delta_1\phi$, $\delta_2\phi$ tangent to $\bar{\mathscr{F}}$. Now, let $\mathcal{H}_\xi$ satisfy

$$\delta\mathcal{H}_\xi = \int_{\partial\mathcal{T}} \delta\boldsymbol{Q} - \xi \cdot \boldsymbol{\theta} + \int_{\partial\mathcal{T}} \xi \cdot \boldsymbol{\Theta} \,. \tag{C.15}$$

This defines a "conserved quantity" up to a constant, which is fixed by requiring that $\mathcal{H}_\xi$ vanishes for a reference solution (Minkowski spacetime). Note that $\mathcal{H}_\xi$ is in general not truly conserved because there is a nonzero flux F_ξ on $\mathcal{B}$ associated with the "conserved quantity". The flux is due to radiation. If we make the reasonable demand that the flux and $\boldsymbol{\Theta}$ vanish if there is no radiation, thus in particular for Minkowski spacetime, one finds

$$\boldsymbol{F}_\xi = \boldsymbol{\Theta}(\phi, \mathcal{L}_\xi\phi) \,. \tag{C.16}$$

This shows that $\boldsymbol{\Theta}$ is directly related to the radiation present in the spacetime and the reason why it has to occur in $\mathcal{H}_\xi$ in the way it does. Note that, additionally, the flux vanishes whenever ξ^a is an exact symmetry, i.e. $\mathcal{L}_\xi\phi = 0$, even if radiation is present.

C.5 BMS symmetry and Bondi Mass in 4 dimensions

Now, we can consider a spacetime which is asymptotically flat at future null infinity $\mathscr{I}^+$. We take $\tilde{g} = \Omega^2 g$, where $\Omega = 0$ on $\mathscr{I}^+$, and $\mathscr{F}$ consists of g. Now, explicit formulas for all forms and quantities defined above can be calculated, e.g., there is the Noether charge

$$Q_{ab}[\xi] = -\frac{1}{16\pi}\epsilon_{abcd}\nabla^c\xi^d \,. \tag{C.17}$$

While it is straightforward to write down the expressions for ω etc. on the physical spacetime, the crucial issue is whether it is possible to extent the presymplectic current 3-form ω continuously to $\mathscr{I}^+$. It turns out that this is indeed possible and it is in general non-vanishing on $\mathscr{I}^+$. Thus, a true Hamiltonian does not exist. Since the infinitesimal asymptotic symmetries are given by the infinitesimal BMS symmetries we want to find the "conserved quantity" $\mathcal{H}_\xi$ for each BMS generator ξ^a and each cross section $\partial\mathcal{T}$ of $\mathscr{I}^+$. Assuming that the vacuum Einstein equations $R_{ab} = 0$ hold, one can define the Bondi news tensor N_{ab} on $\mathscr{I}^+$ and the symplectic potential satisfies

$$\boldsymbol{\Theta} = -\Omega N_{ab}\delta g^{ab}\epsilon^{(3)} \tag{C.18}$$

which defines Θ uniquely and $\epsilon^{(3)}$ is the volume form of dimension 3. Since the representative of the BMS group is not relevant we can choose the Geroch-Winicour gauge $\nabla^a\xi_a = 0$ and thus the "conserved quantity" is

$$\delta\mathcal{H}_\xi = \int_{\partial\mathcal{T}} (\delta\boldsymbol{Q} - \xi \cdot \boldsymbol{\theta}) + \frac{\Omega}{32\pi}\int_{\mathcal{T}} N_{ab}\delta g^{ab}\xi \cdot \epsilon^{(3)} \,, \tag{C.19}$$

which is unique if the Minkowski spacetime is chosen as a reference where $\mathcal{H}_\xi = 0$ to fix the constant. The flux formula is

$$F_\xi = \Theta(g_{ab}, \mathcal{L}_\xi g_{ab}) = -\frac{\Omega}{32\pi} N_{ab} \mathcal{L}_\xi g_{ab} \epsilon^{(3)} . \tag{C.20}$$

To find the "conserved quantity" $\mathcal{H}_\xi$ (and not just its variation) which has the desired properties, [56] compare their results to the classical results of [58, 60, 62] and show that the $\mathcal{H}_\xi$ found by them has to be the solution in the current case, too. This shows that the procedure is in agreement with previous results for $d = 4$.

C.6 BMS symmetry and Bondi Mass in Higher Dimensions

Now, we finally come to the Bondi mass in higher dimensions. The idea of defining asymptotic flatness and null infinity is similar to $d = 4$, see [17] for details. The derivation of $\delta\mathcal{H}_\xi$ is also similar. As before, the crucial issue is to proof that a symplectic current has a finite restriction to $\mathscr{I}^+$ and that there exists a potential Θ for the pullback of symplectic current density to $\mathscr{I}^+$. Again, it is shown that this is possible. The news tensor is defined as follows. Let $n_a = \nabla_a \Omega$, choose any smooth covector field l_a on M such that $l_a l^a = 0$ and $n^a l_a = 1$ at $\mathscr{I}^+$ and set

$$q_{ab} = g_{ab} + 2n_{(a} l_{b)} . \tag{C.21}$$

For $d > 4$ the news tensor on $\mathscr{I}^+$ is defined by

$$N_{ab} = \zeta^\star (\Omega^{(d-4)/2} q_a^m q_b^n S_{mn}) , \tag{C.22}$$

where $\zeta^\star$ is the pullback to $\mathscr{I}^+$. This definition does not hold for $d = 4$. Similarly to $d = 4$, the symplectic potential is defined as

$$\Theta = \frac{1}{32\pi G} \tau^{cd} N_{cd} \epsilon^{(d-1)} , \tag{C.23}$$

$\tau_{ab} = \Omega^{-(d-6)/2} \delta \tilde{g}_{ab}$. The variation of $\mathcal{H}_\xi$ is

$$\delta\mathcal{H}_\xi = \int_{\partial\mathcal{T}} (\delta \boldsymbol{Q} - \xi \cdot \boldsymbol{\theta}) + \frac{1}{32\pi} \int_{\mathcal{T}} N_{cd} \tau^{cd} \xi \cdot \epsilon^{(d-1)} \tag{C.24}$$

which is the same as in $d = 4$, if the correct definition for N_{ab} is used. The flux associated with ξ^a through a segment S of $\mathscr{I}^+$ is

$$F_\xi = \frac{1}{32\pi G} \int_S \chi^{cd} N_{cd} \epsilon^{(d-1)} . \tag{C.25}$$

where $\chi_{ab} = \Omega^{-(d-6)/2}\mathcal{L}_\xi \tilde{g}_{ab}$. Again, the flux vanishes, if the news vanishes. So far, the derivation was basically the same.

Now, consider the special case of "translational" asymptotic symmetries $\xi^a = \alpha n^a - \Omega\nabla^a\alpha$ for some function α. One can show that this is an asymptotic symmetry only if α has the following properties. In the case $d = 4$, α can be any arbitrary function on a given cross section of $\mathscr{I}^+$ which is propagated along the null generator to other cross sections. The corresponding symmetry is called supertranslations (this leads back to the BMS group). However, in $d > 4$ there are only d linearly independent functions α allowed and the associated translational asymptotic symmetries associated with these correspond to the d translational Killing fields in Minkowski spacetime. Thus, there are no additional symmetries due to the asymptotic structure of the spacetime, i.e., there is no analog of the angle-dependent translations in higher dimensions. If $\alpha \geq 0$ the asymptotic translations correspond to the future directed timelike or null translational Killing fields of Minkowski. In this case ($\alpha \geq 0$), the flux formula can be written as

$$F_\xi = -\frac{1}{32\pi G}\int_S \alpha N^{cd}N_{cd}\epsilon^{d-1} \leq 0\,. \tag{C.26}$$

Thus, the energy radiated away is always positive.

Since the goal is to derive an expression for $\mathcal{H}_\xi$ one cannot simply compare $\delta\mathcal{H}_\xi$ to other results (the linkage formalism does not seem to carry over to higher dimensions). Hence, the above equation is not useful anymore and some additional steps are necessary to find $\mathcal{H}_\xi$. Take $\alpha =$ const., simply to make the expressions shorter, i.e., $\xi^a = \alpha n^a$. The idea is to extend the $(d-1)-$form Θ, thus far defined only at $\mathscr{I}^+$, to the entire unphysical spacetime, and then define a new $(d-2)-$form $\boldsymbol{\mu}$ that is related to Θ (but not simply equal to it, $d\boldsymbol{\mu} \neq \Theta$). Since Θ essentially determines the flux this can be used to rewrite down a different expression for the flux. Then, it is shown that

$$H_\xi(\mathcal{B}) = \int_{\mathcal{B}} \boldsymbol{\mu} \tag{C.27}$$

for any cross section $\mathcal{B}$ of $\mathscr{I}^+$. Substituting the definition of $\boldsymbol{\mu}$ this means that the Bondi mass has been found, the result is

$$\mathcal{H}_\xi = \frac{1}{8(d-3)\pi G}\int_{\mathcal{B}} \alpha\Omega^{-(d-4)}\left(\frac{1}{d-2}R_{ab}q^{ac}q^{bd}(\nabla_c l_d)n^e l^f - \Omega^{-1}l^{[e}C^{f]bcd}n_b l_c n_d\right)\epsilon_{ef a_1\ldots a_{d-2}}\,, \tag{C.28}$$

which does not depend on choice of l_a, and where C_{abcd} is the Weyl tensor. Note, that the formula is not correct for $d = 4$. Thus, the Bondi mass is the "conserved quantity" associated to the asymptotic time translation $\xi^a = \alpha n^a$. the integrand of (C.28) is equal to (6.41).

APPENDIX D

Holonomy

We first define some special Riemannian manifolds. Again, we are not very rigorous. Then, after defining the holonomy group, we briefly describe how to characterize Riemannian manifolds in terms of their holonomy group. This can be used to classify all Riemannian manifolds admitting parallel spinors. We state the result in the third section and give some examples.

D.1 Kähler, Calabi, Yau, Sasaki and Einstein

Let (M, g) be a complex Riemannian manifold. A complex manifold "looks locally like" $\mathbb{C}^n$ similarly to how a real manifold "looks locally like" $\mathbb{R}^n$. Let (M, g) be a real manifold then (M, J, g) is a complex manifold where J is globally defined and fiberwise (on each tangent space of the manifold) a linear map $J : TM \to TM$ with $J^2 = -1$, see [20, 113] for a rigorous introduction to complex manifolds and for a discussion of the following definitions. If a smooth manifold M admits a complex structure it must be even-dimensional. If at each point $p \in M$ we have

$$g_p(J_pX, J_pY) = g_p(X, Y) \tag{D.1}$$

for all $X, Y \in T_pM$ then g is called **Hermitian metric** and (M, g) a Hermitian manifold. Given such a metric, we can define the anti-symmetric 2-form

$$\Omega_p(X, Y) := g_p(J_pX, Y) \tag{D.2}$$

which is called **Kähler form**. A **Kähler manifold** is a Hermitian manifold (M, g) whose Kähler form is closed, $d\Omega = 0$. Then g is called Kähler metric. By definition, the Kähler manifold is a symplectic manifold. If (M, g) is a compact Kähler manifold with Ricci flat metric than it is called **Calabi-Yau manifold**. A **Kähler-Einstein metric** is a Riemannian metric that is a Kähler metric and an Einstein metric. The corresponding Kähler-Einstein manifold has constant Ricci curvature (by definition of Einstein metrics) and thus Calabi-Yau manifolds are an example.

For the last notion we want to define we need the following construction. Let (M, g) be a Riemannian metric. Then, the Riemannian cone is defined as the manifold $M \times \mathbb{R}^{>0}$ with metric $t^2 g + dt^2$ where $t \in \mathbb{R}^{>0}$ is a positive real number. Now, let there be a 1-form θ on M. If the 2-form $t^2 d\theta + 2t\theta$ on the cone is a Kähler form and if this makes the cone a Kähler manifold then M is called **Sasakian manifold**. If the cone is additionally Ricci-flat then the manifold is called **Sasaki-Einstein**. If the cone is hyperkähler (see below) then M is called **3-Sasakian**. Since Kähler manifolds are always even dimensional, Sasakian manifolds are always odd dimensional.

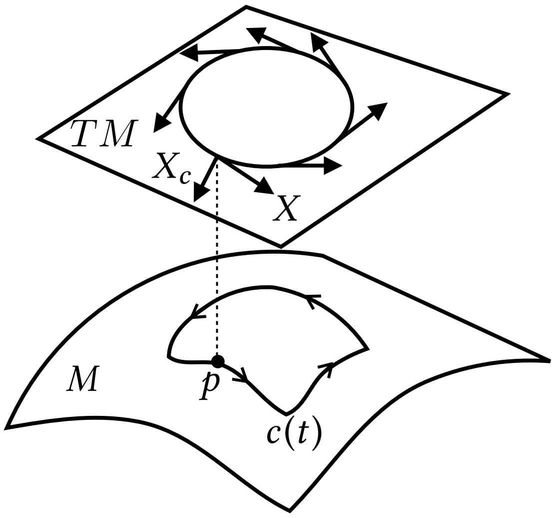


Figure D.1: A loop on M at p lifts to a curve on TM and parallel transport of a vector around the loop is possible. The vector is initially X but after transport along the loop it is X_c and this induces a linear transformation on T_pM.

D.2 Holonomy Group

A Holonomy group can be defined very generally for a connection on a principal $G-$bundle, but we will only consider the case of a tangent space and the Levi-Civita connection. The idea is to capture some aspects of the curvature/geometry of the manifold in a group (via parallel transport) thereby enabling the use of group theory to describe the geometry.

Let (M, g) be an $n-$dimensional Riemannian manifold with Levi-Civita connection ∇. Let p be a point in (M, g), let $\{c(t)|0 \leq t \leq 1,\ c(0) = c(1) = p\}$ be the set of closed loops at p and let $X \in T_pM$. By parallel transporting X once along $c(t)$ we have a new vector $X_c \in T_pM$, see Fig. D.1. Thus, parallel transport along $c(t)$ with connection ∇ induce a linear transformation $\Pi_c : T_pM \to T_pM$. The set of these transformations is called **holonomy group** at p, denoted by $H(p)$, since it can be shown that there is a group structure on this space. See also [20, 26]. If M is connected (as we always assume) the holonomy groups of two points p, q of M are related by conjugation and are thus isomorphic. Hence, it is not necessary to specify the base point and we simply write H. Since the parallel transport preserves the length of a vector, that is $g(X, X) = g(\Pi_c X, \Pi_c X)$, H is a subgroup of $SO(n)$ (since we assume that M is orientable). It is then possible to reduce the classification of Riemannian holonomy groups to representation theory on T_pM. The classification was largely done by Berger [118]. The result is that if a Riemannian manifold is irreducible (the universal cover is not a Riemannian product) and locally non-symmetric ($\nabla R \neq 0$)[1] then the identity component of H must be one of the following ([26])

dim M	H	Geometry
n	$SO(n)$	Generic
$2m$	$U(m)$	Kähler
$2m$	$SU(m)$	Calabi-Yau
$4m$	$Sp(m)$	hyperkähler
$4m$	$Sp(m) \cdot Sp(1) = Sp_1 \times Sp_m / \mathbb{Z}_2$	quaternionic Kähler
7	G_2	exceptional
8	$Spin(7)$	exceptional

Here, $Sp(m) \cdot Sp(1)$ is the image of $Sp_1 \times Sp_m \in \mathrm{Spin}(4m)$ under the map $\mathrm{Spin}(4m) \to SO(4m)$. G_2 is a Lie group which can be defined as the automorphism group of the octonions. A **quaternionic Kähler manifold** is a special kind of Kähler manifold, defined by this holonomy group, and a **hyperkähler manifold** is a Ricci-flat quaternionic Kähler manifold and thus a special kind of a Calabi-Yau manifold. To explain why those are the holonomy groups of the corresponding geometry one can look at the additional structure defined in each case listed in "Geometry" and see that the holonomy group is exactly such that it leaves the defining structure invariant. For example, the holonomy group of a Kähler manifold is contained in $U(m)$, because this group is the subgroup of $O(n)$ which preserves J under parallel transport and thus the defining structure of a Kähler manifold is preserved by $U(m)$.

D.3 Manifolds Admitting Parallel or Killing Spinors

Assuming that a spin manifold admits a Killing spinor yields rather strong restrictions on the possible geometries of the manifold. They can be classified using the corresponding holonomy group. First, we look at the case of parallel spinors and afterwards at Killing spinors with $\lambda \neq 0$. A spin manifold admits parallel spinors only if it is Ricci flat. [33] showed that an irreducible, simply-connected Riemannian spin manifold spin manifolds admits a (non-trivial) parallel spinor only if its holonomy group appears in the following table

dim M	H	Geometry
2m	$SU(m) \subset SO(2m)$	Calabi-Yau
4m	$Sp(m) \subset SO(2m)$	hyperkähler
7	$G_2 \subset SO(7)$	exceptional
8	$Spin(7) \subset SO(8)$	exceptional

[1]The classification of locally symmetric spaces is due to É. Cartan but we will not need this case.

where $m \geq 2$ and N is the dimension of the space of parallel spinors. See section 5 in [114] for examples of manifolds which have these holonomies. We will now turn to Killing spinors. It can be shown that the case of Killing spinors on M can be reduced to the case of parallel spinors on the cone over M, see e.g. [Theorem 14.2.1 in 113]. This was used by Bär in [34] to show that a complete simply-connected Riemannian spin manifold (M, g) admits non-trivial Killing spinors if and only if one of the following possibilities is present

dim M	$H(\bar{g})$	Geometry
n	id	Sphere
$4m+1$	$SU(2m+1)$	Sasaki-Einstein
$4m+3$	$SU(2m+2)$	Sasaki-Einstein
$4m+3$	$Sp(m+1)$	3-Sasakian
7	$Spin(7)$	exceptional
6	G_2	exceptional

where $m \geq 1$, $n > 1$, and $\bar{g}$ is the metric on the Riemannian cone (which is necessarily Ricci flat since the cone admits parallel spinors). $S^2 \times S^3$ is an example of an Sasaki-Einstein manifold in 5 dimensions. Examples of the 6-dimensional case are given in [Theorem 14.3.12 in 113], they are $SU(2) \times SU(2)$, $SU(3)/T^2$ and $\mathbb{C}P^3$. For a discussion of the 7−dimensional case see [113], there are hundreds of examples and the classification is not yet complete. Some examples are ([34]) $SO(5)/SO(3)$, the squashed 7−sphere and Aloff-Wallach manifolds $N_{k,l} = SU(3)/S^1$ where $(k, l) \neq (1, 1)$ and the inclusion $S^1 \to SU(3)$ is given by $z \to \operatorname{diag}\left(z^k, z^l, z^{-l-k}\right)$. Furthermore, examples of 3-Sasakian 7−dimensional spaces are ([Theorem 13.4.6 in 113]) $Sp(2)/Sp(1)$, $Sp(2)/(Sp(1) \times \mathbb{Z}_2)$, $SU(3)/(SU(1) \times U(1))$. Some examples of Sasaki-Einstein 7−dimensional spaces are given in [Proposition 11.7.2 in 113].

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