

Vacuum Polarization of the Charged Klein-Gordon Field

Bachelor Thesis

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Statutory Declaration

I declare that I have written this bachelor thesis by myself, and have not used any other sources than those stated. I clearly stated, when statements or thoughts from literature or any other sources were used.

Signature, Date, Location

Abstract

A charged, massive Klein-Gordon field on a finite interval in $1 + 1$ dimensions in a static, external electric field was considered. A mode decomposition was performed and the modes were calculated explicitly in first order perturbation theory. As a special case, the charge density for the massless Klein-Gordon field was calculated in first order perturbation theory for Dirichlet boundary conditions using Hadamard point-splitting. It could be confirmed that this method yields qualitative different results to the ones obtained by the summation of modes method given by Ambjørn and Wolfram [1], similar to what was found for the Dirac field in [2]. Finally, the vacuum polarization for the massive case was calculated numerically up to first order in the electric field for Dirichlet and Neumann boundary conditions. The charge density was found to vanish on the boundary of the interval for Dirichlet boundary conditions. It was further found to screen the external field for both boundary conditions in contrast to the anti-screening behaviour claimed in [1] for Neumann boundary conditions in the massless limit.

Chapter 1

Introduction

The free Klein-Gordon theory is probably the simplest field theory. The (real) Klein-Gordon field ϕ obeys the free *Klein-Gordon equation*:

$$(\partial_t^2 - \nabla^2 + m^2)\phi(x) = 0 \quad (1.1)$$

With just a few changes, the theory can be used to describe electromagnetic phenomena: One passes to a complex or *charged* scalar field and the usual derivative is replaced by the *covariant derivative*

$$D_\mu = \partial_\mu + ieA_\mu . \quad (1.2)$$

One then speaks about a minimal coupling between the scalar field ϕ and the electromagnetic field, described by the vector potential A_μ . The field responds to the electromagnetic field by a re-distribution of charge and one speaks about a *vacuum polarization*.¹ In this work we will consider the complex Klein-Gordon field minimally coupled with a static, external electric field on a finite interval in $1 + 1$ dimensions, which belongs to what is sometimes called scalar quantum electrodynamics (cf. [3]). For textbooks on this topic and quantum field theory in general we further refer to [4, 5, 3].

In Lagrangian field theory, one easily finds the following expression for the charge density of a Klein-Gordon field:

$$\rho(x) = ie (\phi^*(x) (D_0\phi(x)) - (D_0\phi(x))^* \phi(x)) \quad (1.3)$$

It is however not trivial to make sense of this expression after quantization of the fields, as products of quantum fields evaluated at the same point are not well defined. One method, which we will refer to as the *summation of modes method*, was employed by Ambjørn and Wolfram in [1] to calculate the charge density of the Klein-Gordon field due to a constant, external electric field on a finite interval in $1 + 1$ dimensions. In this method, a mode decomposition is first performed by seeking solutions to the Klein-Gordon equation which have a harmonic time-dependence $\sim e^{-i\Omega_n t}$. One then finds modes with $\Omega_n > 0$, so-called *positive-frequency modes*, as well as *negative frequency modes* with $\Omega_n < 0$. Subsequently, one tries to calculate the contribution to the charge density from each mode separately, ρ_n , and then one pairs each positive frequency mode with its corresponding negative frequency mode to form the so-called induced

¹We will here also refer to this interchangeably as the charge density.

charge density associated with each mode ρ_n^I (cf. [1]). The total charge density is then calculated by summing up all of these contributions according to:

$$\rho(x) = \sum_n \rho_n^I(x) \quad (1.4)$$

It was however shown in [2] that the summation of modes method leads to incorrect results for the Dirac field in an external electric field in 1+1 dimensions. There, the charge density was calculated via a renormalization prescription which we shall refer to as the *Hadamard point-splitting procedure*. One evaluates (1.3) by first calculating the product of fields at different points via the so-called *two-point function*

$$\omega_2 \phi \phi^*(x, y) := \langle 0 | \phi(x) \phi^*(y) | 0 \rangle . \quad (1.5)$$

The singular part of the two-point function is then subtracted by assuming that the two-point function is of Hadamard form and the resulting function is used to calculate the charge density by taking the coinciding-point limit $y \rightarrow x$. The properties of such *Hadamard states* on curved spacetimes have been investigated by many authors. In [6] it was shown that if the two-point function is of Hadamard form in an open neighbourhood of a Cauchy surface, then it is of Hadamard form everywhere. Furthermore, it was e.g. shown in [7] that the ground state of the Klein-Gordon field in a static electric field is of Hadamard form if the external potential is not too strong. Radzikowski [8] gave a mathematically useful definition of the Hadamard condition using *micro-local analysis*. Further motivation for the use of Hadamard states in quantum field theory on curved spacetimes was e.g. given in [9].

The aim of this work is to calculate the vacuum polarization of a Klein-Gordon field on a finite interval in 1 + 1 dimensions via the Hadamard point-splitting procedure and to compare the results to those published by Ambjørn and Wolfram for the massless case (cf. [1]). They found that the vacuum polarization shows a screening behaviour for Dirichlet boundary conditions, which is non-zero on the spatial boundary. For Neumann boundary conditions, they found an anti-screening behaviour within the external field approximation. As we will see, the Hadamard point-splitting procedure yields a screening behaviour for both Dirichlet and Neumann boundary conditions and that the vacuum polarization vanishes on the boundary of the interval for Dirichlet boundary conditions.

In chapter 2 we start by reviewing some important concepts, which are needed for the further analysis. Readers who are familiar with those theoretical preliminaries may want to skip this chapter. In chapter 3 we investigate some properties of the *massive* Klein-Gordon field on a finite interval in 1 + 1 dimensions with Dirichlet and Neumann boundary conditions, similar to the treatment given in [1] for the massless field. For instance, the modes are found analytically and a Krein space description of the problem is introduced. In chapter 4 the modes are calculated explicitly in first order perturbation theory for Dirichlet and Neumann boundary conditions. In chapter 5 the charge density for the massless field subject to Dirichlet boundary conditions is calculated explicitly using the summation of modes method and the Hadamard point-splitting procedure. The charge density for the massive case is further calculated numerically in first order perturbation theory. Finally, the results are summarised and discussed in 6.

1.1 Preface

This bachelor thesis was written as part of my studies at the International Physics Studies Program at the University of Leipzig. During the preceding module "Project-Oriented Course" I verified some of the results presented by Ambjørn and Wolfram in [1] for the massless case. During that time, parts of section 2.1 to 2.4, 3.4 and 5.1 were considered, which are included here as they are required to understand the massive case. These were, however, corrected, generalized or otherwise substantially modified.

I would like to thank Prof. Dr. Hollands for being my supervisor for this thesis and Prof. Dr. Bordag for being the second referee. I would also like to thank Dr. Zahn for all the helpful discussions and for giving me an insight into this topic.

1.2 Notations and Conventions

Throughout this thesis we set $\hbar = c = 1$. The partial derivative with respect to a variable x is denoted by $\frac{\partial}{\partial x}$ or ∂_x . We work on flat Minkowski space, such that the zeroth component is the time component and we use the signature² $(+1, -1, -1, -1)$. Greek letters usually denote the components of vectors on the Minkowski space, where as the indices i or j usually refer to the spatial components. We use the summation convention that double appearing indices are summer over, such that i.e. $D_\mu^2 = D_\mu D^\mu = \sum_{\mu,\nu} \eta_{\mu\nu} D^\mu D^\nu$, where η is the metric tensor.

Further, the time component of a vector x is often denoted by x^0 and the spatial component by x^1 . The first argument of a function is usually denoted by x and the second one by x' or y , where e.g. x refers to a vector, if it is not clear from the context that only the spatial component is meant.

²When in $1 + 1$ dimensions we, of course, use $(+1, -1)$.

Chapter 2

Theoretical Preliminaries

2.1 The Euler-Lagrange Equations of Motion

We will here briefly discuss the Euler-Lagrange equations of motion in Lagrangian field theory, which will be important in our further discussion of the Klein-Gordon field. For a full derivation we refer to e.g. [3]. In Lagrangian mechanics, the action S is related to the Lagrangian (denoted here by L'), by (cf. [3, p. 15-16])

$$S = \int L' dt . \quad (2.1)$$

There, the Lagrangian is a function of generalized coordinates $\{q_i\}_{i=1}^N$ and their time derivatives. In passing to a field theoretical description, one passes to a Lagrangian density¹ (here denoted by L), which depends on at least one field and its derivatives. The action is then given by²

$$S = \int L(\phi, \partial_\mu \phi) d^d x \quad (2.2)$$

As in classical mechanics, one here also employs the principle of least action, i.e. the variation of the action should be stationary, $\delta S = 0$. By assuming no variations on the boundary, one can rewrite this condition as a condition on the field and one obtains the Euler-Lagrange equations of motion (cf. [3])

$$\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} = 0 . \quad (2.3)$$

2.2 Noether's Theorem

Noether's theorem shows how symmetries in the Lagrangian relates to conserved quantities. We will later use this theorem to find an expression for the electromagnetic current of the Klein-Gordon field.

The theorem may be stated in the following way:³ If, a transformation of the field⁴,

¹We will, however, refer to L as the Lagrangian in our further discussions.

²cf. [3]. d stands for the dimension (including the time dimension) of the system.

³cf. [3, p.17-18]. Again, for readers who are interested in a full derivation we refer to textbooks such as [3].

⁴The field is assumed to satisfy the Euler-Lagrange equations of motion.

which infinitesimally may be written as

$$\phi(x) \rightarrow \phi(x) + \alpha \delta \phi(x) , \quad (2.4)$$

only changes the Lagrangian up to a divergence term J^μ

$$L \rightarrow L + \alpha \partial_\mu J^\mu(x) , \quad (2.5)$$

then the following relation holds:⁵

$$\partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi)} \delta \phi - J^\mu \right) = 0 . \quad (2.6)$$

The above expression in the parentheses is thus a conserved current. The spatial integral of the zeroth component of this current is constant in time, and is sometimes referred to as a conserved charge (cf. [3]).

2.3 Indefinite Inner Product Spaces and Krein Spaces

We will here introduce the concept of Krein spaces similar to the way it's done in [10]. But, before defining what a Krein space is, we will introduce some notations. Consider the infinite-dimensional, complex vector space K equipped with a hermitian, sesquilinear form

$$\langle \cdot | \cdot \rangle : K \times K \rightarrow \mathbb{C} . \quad (2.7)$$

That is, $\langle \cdot | \cdot \rangle$ has the following properties:

- $\forall x_1, x_2, y_1, y_2 \in K, a \in \mathbb{C}$
- 1. $\langle x_1 + x_2 | y_1 + y_2 \rangle = \langle x_1 | y_1 \rangle + \langle x_1 | y_2 \rangle + \langle x_2 | y_1 \rangle + \langle x_2 | y_2 \rangle$
- 2. $\langle x_1 | x_2 \rangle = \langle x_2 | x_1 \rangle^*$
- 3. $\langle x_1 | ax_2 \rangle = a \langle x_1 | x_2 \rangle$.

As in [10], we may categorize the elements $x \in K$ after the value of the expression $\langle x | x \rangle$ in positive, neutral and negative elements. The set of all the *neutral* elements of K we will denote by $B_{\langle \cdot | \cdot \rangle}^0$, which is thus given by $B_{\langle \cdot | \cdot \rangle}^0 = \{x \in K | \langle x | x \rangle = 0\}$. One may similarly define $B_{\langle \cdot | \cdot \rangle}^{00}$ to be given by all the neutral elements except the zero element, i.e. $B_{\langle \cdot | \cdot \rangle}^{00} = B_{\langle \cdot | \cdot \rangle}^0 \setminus \{0_K\}$. The positive elements are denoted by $B_{\langle \cdot | \cdot \rangle}^+ = \{x \in K | \langle x, x \rangle > 0\}$. However, as $B_{\langle \cdot | \cdot \rangle}^+$ is not a subspace of K (it does not contain the zero element), we may define $B_{\langle \cdot | \cdot \rangle}^{++} = B_{\langle \cdot | \cdot \rangle}^+ \cup \{0_K\}$. Sets $B_{\langle \cdot | \cdot \rangle}^-$ and $B_{\langle \cdot | \cdot \rangle}^{--}$ are then defined analogously for the negative elements. Of course, when it is clear which hermitian sesquilinear form one is referring to, one might drop the subscripts $\langle \cdot | \cdot \rangle$.

We may now state what we mean with e.g. an *indefinite* inner product space: K and its hermitian sesquilinear form $\langle \cdot | \cdot \rangle$ are called an indefinite inner product space, if K has elements in both $B_{\langle \cdot | \cdot \rangle}^{++}$ and $B_{\langle \cdot | \cdot \rangle}^{--}$ (cf. [10, p. 4]). One may then refer to $\langle \cdot | \cdot \rangle$ as an indefinite inner product.

A *Krein* space is a special type of indefinite inner product space, which one might call decomposable and non-degenerate (cf. [10, p. 100]). Precisely, we will call a vector

⁵cf. [3, p.17-18]. If the transformation changes multiple fields, then the first term in the parentheses should be replaced by a sum over the appropriate fields.

space K and a hermitian sesquilinear form $\langle \cdot | \cdot \rangle$ a Krein space $(K, \langle \cdot | \cdot \rangle)$ if K can be decomposed into *intrinsically complete* subspaces K^+ , K^- , such that (cf. [10, p. 100])

$$K = K^+ \oplus K^- : \quad K^+ \subset B^{++}, K^- \subset B^{--} . \quad (2.8)$$

The term intrinsically complete, means that K^+ and K^- should be complete with respect to the norms $\|x\|_{\pm} = \sqrt{\pm \langle x | x \rangle}$ respectively. This means that a Krein space can be spanned by vectors which are not neutral, which is of crucial importance when one wants to do perturbative calculations.

2.4 Perturbation Theory on Krein Spaces

Consider the operator H on the Krein space $(K, \langle \cdot | \cdot \rangle)$, given by

$$H = H_0 + \lambda H_1 , \quad (2.9)$$

where H, H_0, H_1 are hermitian operators with respect to $\langle \cdot | \cdot \rangle$, and $\lambda \in (0, \infty)$. We shall think of λ as a small, positive, perturbation parameter. Assume that H_0 has a countable set⁶ of eigenvectors $\{\Psi_n\}_{n \in I}$ with *non-degenerate* eigenvalues Ω_n , which forms an *orthogonal* basis in K and that $\exists f : I \rightarrow \mathbb{C} \setminus \{0\}$, such that Ψ_n can be chosen to have the following normalization:

$$\langle \Psi_n | \Psi_n \rangle = f(n) . \quad (2.10)$$

That is, *all* of the eigenvectors Ψ_n are not neutral. Having clarified the setup, we now turn to the problem of finding approximate solutions to the eigenvalue problem

$$H\Phi = \Omega\Phi , \quad (2.11)$$

by considering λH_1 to be a small perturbation of H_0 .⁷ For this reason we assume that each Φ_n , and Ω_n have a series expansion in λ

$$\begin{aligned} \Phi_n &= \sum_{j=0}^{\infty} \lambda^j \Phi_n^{(j)} \\ \Omega_n &= \sum_{j=0}^{\infty} \Omega_n^{(j)} \lambda^j , \end{aligned} \quad (2.12)$$

where $\Phi_n^{(0)} = \Psi_n$. We impose the normalization that Φ_n has the normalization $f(n)$ to first order in λ . Thus⁸,

$$\begin{aligned} \langle \Phi_n | \Phi_n \rangle &= \left\langle \Phi_n^{(0)} \left| \Phi_n^{(0)} \right. \right\rangle + \lambda \left(\left\langle \Phi_n^{(1)} \left| \Phi_n^{(0)} \right. \right\rangle + \left\langle \Phi_n^{(0)} \left| \Phi_n^{(1)} \right. \right\rangle \right) + O(\lambda^2) \\ &= f(n) + 2\lambda \operatorname{Re} \left\langle \Phi_n^{(0)} \left| \Phi_n^{(1)} \right. \right\rangle + O(\lambda^2) , \end{aligned} \quad (2.13)$$

wherefore we require

$$\left\langle \Phi_n^{(0)} \left| \Phi_n^{(1)} \right. \right\rangle = 0 . \quad (2.14)$$

⁶The index set is here denoted by I .

⁷The derivation is very similar to the usual derivations of first order perturbation theory given in textbooks on quantum mechanics (see [11]), only taking the special normalization into account.

⁸ $O(\lambda^2)$ means terms that approach 0 as λ^2 or faster as $\lambda \rightarrow 0^+$

Inserting the expansion, together with (2.9), into (2.11), and equating the terms of zeroth and first order in λ , yields

$$\begin{aligned} H_0 \Phi_n^{(0)} &= \Omega_n^{(0)} \Phi_n^{(0)} \\ H_0 \Phi_n^{(1)} + H_1 \Phi_n^{(0)} &= \Omega_n^{(0)} \Phi_n^{(1)} + \Omega_n^{(1)} \Phi_n^{(0)} . \end{aligned} \quad (2.15)$$

The first equation is simply the unperturbed equation. By taking the inner product on both sides with $\Phi_n^{(0)}$ of the second equation one easily finds

$$\Omega_n^{(1)} = \frac{1}{f(n)} \left\langle \Phi_n^{(0)} \left| H_1 \Phi_n^{(0)} \right. \right\rangle . \quad (2.16)$$

Equation (2.16) gives the first order perturbative correction to the n^{th} eigenvalue. We proceed to find the first order corrections to the eigenvectors. In doing so we expand $\Phi_n^{(1)}$ in terms of the basis $\{\Phi_k^{(0)}\}$ (cf. [11])

$$\Phi_n^{(1)} = \sum_{m \in I \setminus \{n\}} c_m \Phi_m^{(0)} . \quad (2.17)$$

The reason that we don't sum over $k = n$ is that $\Phi_n^{(1)}$ and $\Phi_n^{(0)}$ are orthogonal due to our imposed normalization (Equation (2.14)). By plugging this into the second equation in (2.15) and taking the inner product with $\Phi_k^{(0)}$, $k \neq n$, one easily finds

$$c_k = \frac{1}{f(k)} \frac{\left\langle \Phi_k^{(0)} \left| H_1 \Phi_n^{(0)} \right. \right\rangle}{\Omega_n^{(0)} - \Omega_k^{(0)}} , \quad (2.18)$$

by using the hermiticity of H_0 and the orthogonality of the basis vectors. Note that we could only divide by $\Omega_n^{(0)} - \Omega_k^{(0)}$ because we assumed that the eigenvalues are non-degenerate. Hence, we arrive at the final result for the first order correction to the eigenvectors

$$\Phi_n^{(1)} = \sum_{k \in I \setminus \{n\}} \frac{1}{\langle \Phi_k^{(0)} | \Phi_k^{(0)} \rangle} \frac{\left\langle \Phi_k^{(0)} \left| H_1 \Phi_n^{(0)} \right. \right\rangle}{\Omega_n^{(0)} - \Omega_k^{(0)}} \Phi_k^{(0)} . \quad (2.19)$$

One immediately recognizes that this can not be used if any of the basis vectors $\Phi_k^{(0)}$ is neutral.

2.5 The Hadamard Condition

When calculating the vacuum polarization we will assume that the two-point function has a singularity of Hadamard form.⁹ The precise singularity structure of such functions depend on the dimension.¹⁰ In 1 + 1 dimensions, we will say that a function f has a singularity structure of Hadamard form, if (cf. [6, 2, 14])

$$\begin{aligned} f(x, x') &= \lim_{\varepsilon \rightarrow 0^+} \left[\frac{u(x, x')}{(x - x')_\varepsilon^2} + v(x, x') \ln(-(x - x')_\varepsilon^2) + w(x, x') \right] \\ &\equiv \lim_{\varepsilon \rightarrow 0^+} [h_\varepsilon + w(x, x')] , \end{aligned} \quad (2.20)$$

⁹Hadamard studied such functions when seeking solutions of second order partial differential equations [6, 12].

¹⁰See [13] for a treatment of Hadamard expansions in quantum field theory in different dimensions.

where $(x - x')_\varepsilon^2 = (x - x')^2 - i\varepsilon(x - x')^0$. u , v and w should further be smooth functions and $v(x, x')$ and $w(x, x')$ should possess series expansions in $(x - x')^2$, such that (cf. [6])

$$\begin{aligned} v(x, x') &= \sum_n v_n(x, x')(x - x')^{2n} \\ w(x, x') &= \sum_n w_n(x, x')(x - x')^{2n} . \end{aligned} \quad (2.21)$$

If one now requires $f(x, x')$ to be a solution of some differential equation (i.e. the Klein-Gordon equation if f is the two-point function) one may determine u and v_n by the Hadamard recursion relations (cf. [6, 12]), which we will, however, not discuss any further here. The term h_ε in (2.20) specifies the singular form of f in the coinciding-point limit and will here be referred to as the *parametrix*. The third term, $w(x, x')$, is the smooth part of f .

The reason that the Hadamard form is of interest in quantum field theory in external potentials is that the two-point functions of many states have this form, e.g. it was shown in [7] that the ground state of the Klein-Gordon field is a Hadamard state for sufficiently weak external potentials.

The properties of two-point functions of Hadamard form have further been studied by Radzikowski in [8]. Two important conditions¹¹ on the two-point function are that it should satisfy the Klein-Gordon equation and that (cf. [8])

$$\omega_2 \phi \phi^*(x, y) - \omega_2 \phi^* \phi(x, y) = i\Delta . \quad (2.22)$$

Here $\Delta = \Delta^+ - \Delta^-$, where Δ^\pm are the advanced/retarded fundamental solutions of the Klein-Gordon operator. These solve the inhomogeneous Klein-Gordon equation (with the δ -function as inhomogeneity) and are non-zero only in the casual future/past (cf. [7, 6]).

2.6 Parametrix of the Two-point Function

To find the parametrix of the Klein-Gordon operator, one can proceed analogously to [2], where the parametrix of the Dirac operator was constructed.

An operator is said to be *normally hyperbolic* if its principal symbol is given by the metric, that is, a normally hyperbolic operator P takes on the form (cf. [15, 7])

$$P = \partial_\mu \partial^\mu + A_\mu(x) \partial^\mu + B(x) , \quad (2.23)$$

where $A_\mu(x)$ and $B(x)$ are smooth functions. We especially note that the Klein-Gordon operator $D_\mu^2 + m^2$ is a normally hyperbolic operator. The fundamental solutions of normally hyperbolic operators can be constructed formally and uniquely out of the so-called advanced and retarded *Riesz distributions*¹² $R_j^\pm(x, x')$. As in [15], one formally makes the ansatz

$$\Delta^\pm(x, x') = \sum_{k=0}^{\infty} V_k(x, x') R_{2+2k}^\pm . \quad (2.24)$$

¹¹Further conditions are given by Radzikowski in terms of micro-local analysis, which will not be discussed any further here.

¹²See [15] for an extensive treatment of this subject.

By applying the operator P onto the ansatz and using the properties of the Riesz distributions one can derive the transport equation (cf. [2, 15]):

$$(x - x')^\mu D_\mu V_k(x, x') + k V_k(x, x') = -k P V_{k-1}(x, x') \quad (2.25)$$

The transport equation imposes a condition on the coefficients V_k , in order for (2.24) to be a fundamental solution. The coefficients V_k are sometimes referred to as *Hadamard coefficients* (see [15]). V_0 will be of special importance to us, which we will refer to as the *parallel transport*. With the transport equation, one especially finds that the Hadamard coefficients are the same for both the advanced and the retarded solutions (cf. [15]). The transport equation allows one to recursively determine the Hadamard coefficients. By putting $k = 0$ in (2.25) and using the condition $V_0(x, x) = 1$ one can derive the formula (cf. [2]):

$$V_0(x, x') = \exp \left(-ie \int_0^1 A_\mu(x' + t(x - x'))(x - x')^\mu dt \right) \quad (2.26)$$

This formula will be of crucial importance when calculating the charge density using the Hadamard point-splitting procedure, as we will see later.

The parametrix for the Klein-Gordon operator can now be constructed as (cf. [2]):

$$h^\pm(x, x') = \frac{1}{2\pi} \sum_{k=0}^{\infty} V_k(x, x') T_{2+2k}^\pm(x, x') , \quad (2.27)$$

where T_j are distributions that are depending on the dimension.¹³ In 1+1 dimensions, they are given by (cf. [2])

$$T_{2k}^\pm(x, x') = \frac{-1}{2^{2k-1}(k-1)!} (x - x')^{2(k-1)} \log \frac{-(x - x')_{\pm\epsilon}^2}{\Lambda^2} , \quad (2.28)$$

where $\Lambda \in \mathbb{R}^+$ can be thought of as a length scale. One can show that (2.27) satisfy the appropriate conditions set up by Radzikowski (cf. [2, 8]). What one needs to do in order to find the explicit form of the parametrix is to determine the Hadamard coefficients V_k . Fortunately, one does not need to calculate all of them, as the T_j 's vanish increasingly fast in the coinciding point limit as one can see by inspecting (2.28). In our case, as the expression for the charge density of the Klein-Gordon field only contains first-order derivatives, only the parallel transport V_0 is needed to calculate the parametrix, since the derivative of higher order terms vanish in the coinciding point limit.

¹³Especially, they satisfy $T_j^+ - T_j^- = 2\pi i (R_j^+ - R_j^-)$, $j \in \mathbb{N}_0$ (cf. [2]).

Chapter 3

Properties of the Klein-Gordon Field Minimally Coupled with an Electromagnetic Potential

3.1 Equations of Motion and Definition of the Current in Lagrangian Field Theory

The Lagrangian for a Klein-Gordon field minimally coupled with an electromagnetic field is given by¹

$$L = |D_\mu \phi|^2 - m^2 |\phi|^2 - \frac{1}{4} F_{\mu\nu}^2 - j_\mu^{ext} A^\mu , \quad (3.1)$$

where D_μ denotes the covariant derivative and $F_{\mu\nu}$ the electromagnetic field strength tensor

$$\begin{aligned} D_\mu &= \partial_\mu + ie A_\mu \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu . \end{aligned} \quad (3.2)$$

j_μ^{ext} denotes an external electromagnetic current. The notation $F_{\mu\nu}^2$ means a contraction over both indices with itself, i.e. $F_{\mu\nu}^2 = F_{\mu\nu} F^{\mu\nu}$.

The Euler-Lagrange equations of motion for ϕ , ϕ^* and A_μ can be straightforwardly computed according to (2.3) and one finds

$$\begin{aligned} (D_\mu D^\mu + m^2) \phi &= 0 \\ \partial_\mu F^{\mu\nu} &= j_{ext}^\nu + ie (\phi^* D^\mu \phi - \phi (D^\mu \phi)^*) . \end{aligned} \quad (3.3)$$

The first equation is simply the Klein-Gordon equation. The equation for ϕ similarly yields the complex conjugate of the Klein-Gordon equation. The second equation is related to the electromagnetic current. By noting that the Lagrangian is invariant under global phase transformations

$$\phi \rightarrow e^{i\alpha} \phi , \quad (3.4)$$

one can easily derive using Noether's theorem (2.6) a conserved current:

$$j^\mu = i (\phi (D^\mu \phi)^* - \phi^* D^\mu \phi) \quad (3.5)$$

¹cf. [1]. We take this Lagrangian as a starting point. The aim of this section is merely to use this Lagrangian to define an expression for the electromagnetic current.

By comparing this result to the above Euler-Lagrange equation for A_μ we identify the conserved electromagnetic current as (cf. [1])

$$J^\mu = -ej^\mu = ie(\phi^* D^\mu \phi - \phi (D^\mu \phi)^*) . \quad (3.6)$$

By introducing the conjugate momentum π to the field

$$\pi := \frac{\partial L}{\partial(\partial_0 \phi)} = (\partial^0 \phi^* - ieA^0 \phi^*) = (D_0 \phi)^* , \quad (3.7)$$

one may especially define the charge density as (cf. [1])

$$\rho(x) = J_0 = ie(\phi^*(x)\pi^*(x) - \phi(x)\pi(x)) . \quad (3.8)$$

3.2 Explicit Solution of the Klein-Gordon Equation on a Finite Interval

We are now considering a massive Klein-Gordon field coupled with a static electric field E on the finite interval $[0, a]$ in 1+1 dimensions. The spatial coordinate is denoted by Z . We start with the general equations of motion² (3.3)

$$\left(\frac{d}{dt} + ieA_0\right)^2 \phi - \frac{d^2}{dZ^2} \phi + m^2 \phi = 0 \quad (3.9)$$

As in [1], we make the mode ansatz

$$\phi(Z, t) = \phi_n(Z) e^{-i\Omega_n t} , \quad (3.10)$$

such that (3.9) may be written as

$$\left((\Omega_n - eA_0)^2 - m^2 + \frac{d^2}{dZ^2}\right) \phi(Z) = 0 . \quad (3.11)$$

We now introduce the following definitions³:

$$\begin{aligned} z &= \frac{Z}{a} \\ \lambda &= eEa^2 \\ A_0 &= -Ea \left(z - \frac{1}{2} + \alpha\right) \\ \omega_n &= a\Omega_n + \lambda\alpha , \end{aligned} \quad (3.12)$$

It should be noted that z , λ and ω_n are dimensionless quantities in 1+1 dimensions. α is a gauge parameter. The dimensionless coordinate z thus ranges from 0 to 1. ω_n was chosen such that it is a dimensionless, gauge invariant parameter. We also see that the electromagnetic potential A indeed corresponds to a static electric field of

² A_1 is chosen to be 0.

³These are the same as Ambjørn and Wolfram use, except that their definition of A_0 contains an extra factor Ea (cf. [1]). It was chosen not to include it here, such that $\frac{dA_0}{dZ} = -E$

strength E . Upon doing a change of variables from Z to z the derivative transforms in the following way

$$\frac{d}{dZ} = \frac{dz}{dZ} \frac{d}{dz} = \frac{1}{a} \frac{d}{dz} .$$

If we use this transformation law and plug in our definitions (Eq. (3.12)), we get

$$\left[\left(\omega_n + \lambda \left(z - \frac{1}{2} \right) \right)^2 - a^2 m^2 + \frac{d^2}{dz^2} \right] \phi_n(z) = 0 . \quad (3.13)$$

Equation (3.13) is thus the equation of motion for the modes of a massive Klein-Gordon field minimally coupled with a static electric field. To solve this equation analytically we introduce the variable ξ defined by:

$$\xi = C \left(\omega_n + \lambda \left(z - \frac{1}{2} \right) \right) , \quad \frac{d}{dz} = C \lambda \frac{d}{d\xi}$$

C is a constant yet to be chosen. Thus:

$$\begin{aligned} \frac{\xi^2}{C^2} \phi_n - a^2 m^2 \phi_n + C^2 \lambda^2 \frac{d^2}{d\xi^2} \phi_n &= 0 \\ \frac{d^2}{d\xi^2} \phi_n - \left(\frac{-4}{\lambda^2 C^4} \right) \frac{\xi^2}{4} \phi_n - \frac{a^2 m^2}{C^2 \lambda^2} \phi_n &= 0 \end{aligned}$$

Choosing $C = \frac{1+i}{\sqrt{\lambda}}$, this can be written as:

$$\frac{d^2}{d\xi^2} \phi_n + \left[\left(i \frac{a^2 m^2}{2\lambda} - \frac{1}{2} \right) + \frac{1}{2} - \frac{\xi^2}{4} \right] \phi_n = 0 \quad (3.14)$$

This is a Weber⁴ differential equation of order $i \frac{a^2 m^2}{2\lambda} - \frac{1}{2}$. The solution is given in terms of the parabolic cylinder functions

$$\phi_n(z) = a_n D_{\frac{im^2 a^2}{2\lambda} - \frac{1}{2}}(\xi(z)) + b_n D_{-\frac{im^2 a^2}{2\lambda} - \frac{1}{2}}(i\xi(z)) , \quad (3.15)$$

where

$$\xi(z) = \frac{1+i}{\sqrt{\lambda}} \left(\omega_n + \lambda \left(z - \frac{1}{2} \right) \right) . \quad (3.16)$$

a_n , b_n and ω_n are to be found from the boundary conditions and the imposed normalization. We will here only consider Dirichlet ($\phi(0) = \phi(1) = 0$) and Neumann ($\frac{\partial \phi(0)}{\partial z} = \frac{\partial \phi(1)}{\partial z} = 0$) boundary conditions.

As a special case, one can now easily find the modes in the massless limit $m \rightarrow 0$, namely:

$$\phi_{n,m=0}(z) = a_n D_{-\frac{1}{2}}(\xi(z)) + b_n D_{-\frac{1}{2}}(i\xi(z)) , \quad (3.17)$$

It should be noted that this solution differs slightly from the one presented by Ambjørn and Wolfram (cf. [1])

$$\begin{aligned} \phi_n &= a_n D_{-\frac{1}{2}} \left((1+i)\sqrt{2}\sqrt{\lambda} \left(\omega_n + \left(z - \frac{1}{2} \right) \right) \right) \\ &+ b_n D_{-\frac{1}{2}} \left(-(1-i)\sqrt{2}\sqrt{\lambda} \left(\omega_n + \left(z - \frac{1}{2} \right) \right) \right) . \end{aligned} \quad (3.18)$$

⁴The Weber differential equation $\frac{d^2 f(z)}{dz^2} + (n + \frac{1}{2} - \frac{1}{4}z^2)f(z) = 0$ has the two linearly independent solutions $f_1(z) = D_n(z)$ and $f_2(z) = D_{-n-1}(iz)$, where $D_n(z)$ are the parabolic cylinder functions (cf. [16, p. 347]).

Of course, there is an infinite number of choices for the two basis functions that span the solution space, but as the two expressions have different functional dependence on the parameter λ , one may conclude that the solutions are indeed different. This seems to be a mistake of Ambjørn and Wolfram.

3.3 The Krein Space Description

We wish to be able to apply perturbation theory to the equations of motions in a weak, static external electromagnetic field. To do this we will, analogous to Ambjørn and Wolfram, introduce an indefinite inner product $\langle \cdot | \cdot \rangle$ and reformulate the problem on a Krein space. The formalism presented in this section will be applicable to a finite volume in d dimensions.

We start by introducing the operator H ,⁵ which acts on the space $L^2 \oplus L^2$

$$H = \begin{pmatrix} eA_0 & 0 \\ 0 & eA_0 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ -(-D_i^2 + m^2) & 0 \end{pmatrix} .$$

The operator H enables us to write the equations of motion in a Hamiltonian-like form⁶ (cf. [1])

$$\begin{aligned} i \frac{\partial}{\partial t} \Psi(z, t) &= H \Psi(z, t) \\ \partial^\mu F_{\mu\nu} &= -e \Psi^*(z, t) \sigma_2 \Psi(z, t) , \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} \Psi(z, t) &= \begin{pmatrix} \phi(z, t) \\ \pi^*(z, t) \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \end{aligned}$$

After a mode decomposition

$$\Psi_n(z, t) = \Psi_n(z) e^{-i\Omega_n t} . \tag{3.20}$$

one finds the following equation as a straightforward consequence:

$$H \Psi_n(z) = \Omega_n \Psi_n(z) \tag{3.21}$$

The above statements are shown by first calculating $H\Psi$:

$$\begin{aligned} H\Psi &= \begin{pmatrix} eA_0 & 0 \\ 0 & eA_0 \end{pmatrix} \begin{pmatrix} \phi \\ \pi^* \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ -(-D_i^2 + m^2) & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \pi^* \end{pmatrix} \\ &= \begin{pmatrix} eA_0 \phi + i\pi^* \\ eA_0 \pi^* - i(-D_i^2 + m^2)\phi \end{pmatrix} , \end{aligned}$$

and comparing this to the left-hand side of (3.19)

$$i \frac{\partial}{\partial t} \Psi = i \begin{pmatrix} \frac{\partial}{\partial t} \phi \\ \frac{\partial}{\partial t} \pi^* \end{pmatrix} .$$

⁵cf. [1]. Note that there is a minus sign missing in the paper by Ambjørn and Wolfram. There, the operator was stated as $H = \begin{pmatrix} eA_0 & 0 \\ 0 & eA_0 \end{pmatrix} + i \begin{pmatrix} 0 & 1 \\ -D_i^2 + m^2 & 0 \end{pmatrix}$

⁶An alternative way of doing this was presented in [7].

The first component yields after some rearranging $\pi = (D_0\phi)^*$, which agrees with our previous calculation of the momentum (3.7). Turning to the second component one finds (by using the first equation) $(D_\mu D^\mu + m^2)\phi = 0$, which is just the Klein-Gordon equation. Further, $\partial^\mu F_{\mu\nu} = -e\Psi^*\sigma_2\Psi$ is straightforward to show by calculating the right-hand side and using equation (3.3).

Furthermore, H is hermitian under Dirichlet and Neumann boundary conditions⁷ with respect to the indefinite inner product⁸

$$\langle\Psi_1|\Psi_2\rangle = - \int d^d x \Psi_1^* \sigma_2 \Psi_2 = i \int d^d x (\phi_1^* \pi_2^* - \pi_1 \phi_2) . \quad (3.22)$$

The hermiticity of H is shown by calculating

$$\langle\Psi_1|H\Psi_2\rangle - \langle H\Psi_1|\Psi_2\rangle = \int \left[\phi_2 (D_i^2 \phi_1)^* - \phi_1^* D_i^2 \phi_2 \right] d^d x , \quad (3.23)$$

and using the divergence theorem and the boundary conditions. For Dirichlet and Neumann boundary conditions in 1 + 1 dimensions this reduces to performing partial integration. $(L^2 \oplus L^2, \langle \cdot | \cdot \rangle)$ and H thus satisfy the conditions in section 2.4 and may be used for perturbation theory.

3.4 Quantization of the Klein-Gordon Field

We will here present a quantization procedure, which was also used by Ambjørn and Wolfram in [1]. We start by writing the quantum field as a linear combination of the positive and negative frequency modes with operator-valued coefficients⁹

$$\phi(x, t) = \sum_{m,+} a_m \phi_m^{(+)}(x) e^{-i\Omega_m^{(+)} t} + \sum_{m,-} b_m^\dagger \phi_m^{(-)}(x) e^{-i\Omega_m^{(-)} t} . \quad (3.24)$$

From the relation $\pi = (D_0\phi)^*$, we find

$$\begin{aligned} \pi(x, t) &= \sum_{m,+} a_m^\dagger (i\Omega_m^{(+)} - ieA_0) (\phi_m^{(+)}(x))^* e^{+i\Omega_m^{(+)} t} \\ &+ \sum_{m,-} b_m (i\Omega_m^{(-)} - ieA_0) (\phi_m^{(-)}(x))^* e^{+i\Omega_m^{(-)} t} . \end{aligned} \quad (3.25)$$

We proceed by postulating the equal-time commutation relations¹⁰ (cf. [1])

$$\begin{aligned} [\phi(x, t), \pi(y, t)] &= i\delta(x - y) \\ [\phi(x, t), \phi^\dagger(y, t)] &= 0 \\ [\pi(x, t), \pi^\dagger(y, t)] &= 0 . \end{aligned} \quad (3.26)$$

⁷It is actually hermitian with respect to the more general boundary conditions $(n^i D_i + \chi)\phi = 0$, where χ is a function which is defined on the boundary and n^i an outward normal to the boundary (cf. [1]). $\chi = 0, \infty$ correspond to Dirichlet and Neumann boundary conditions, respectively.

⁸cf. [1]. Here, $\int d^d x$ means integration over the finite, d -dimensional spatial volume under interest.

⁹cf. [1]. The notation $\sum_{m,+}, \sum_{m,-}$ means that the index m sums over the positive frequency modes in the first sum and negative frequency modes in the second one.

¹⁰ $[a, b] := ab - ba$

We want to show that these commutation relations imply

$$\begin{aligned} [a_m, a_n^\dagger] &= \delta_{m,n} \\ [b_m, b_n^\dagger] &= \delta_{m,n} . \end{aligned} \quad (3.27)$$

To show this we turn to the Krein space description.¹¹ This yields

$$\Psi(x, t) = \sum_{m,+} a_m \Psi_m^{(+)}(x) e^{-i\Omega_m^{(+)}t} + \sum_{m,-} b_m^\dagger \Psi_m^{(-)}(x) e^{-i\Omega_m^{(-)}t} . \quad (3.28)$$

We assume that the modes are orthogonal with respect to each other with respect to $\langle \cdot | \cdot \rangle$ and that they have normalization ± 1 . Taking the inner product with $\Psi_m^{(+)}(x)$ and $\Psi_n^{(+)}(x)$, we can solve for a and a^\dagger :¹²

$$\begin{aligned} \langle \Psi_m(x) | \Psi(x, t) \rangle &= a_m e^{-i\Omega_m t} \\ \langle \Psi(x, t) | \Psi_n(x) \rangle &= a_n^\dagger e^{i\Omega_n t} \end{aligned} \quad (3.29)$$

Hence,

$$[a_m, a_n^\dagger] = e^{it(\Omega_m - \Omega_n)} [\langle \Psi_m | \Psi \rangle, \langle \Psi | \Psi_n \rangle] . \quad (3.30)$$

Using the definition of the inner product (Eq. (3.22)), it follows

$$\begin{aligned} \langle \Psi_m | \Psi \rangle \langle \Psi | \Psi_n \rangle &= - \int dx dy \left[\phi_m^*(x) \pi^\dagger(x, t) \phi^\dagger(y, t) \pi_n^*(y, t) - \phi_m^*(x) \pi^\dagger(x, t) \pi(y, t) \phi_n(y) \right. \\ &\quad \left. - \pi_m(x) \phi(x, t) \phi^\dagger(y, t) \pi_n(y) + \pi_m(x) \phi(x, t) \pi(y, t) \phi_n(y) \right] . \end{aligned}$$

Using the commutation relations (3.27), one finds

$$\begin{aligned} [a_m, a_n^\dagger] &= -e^{it(\Omega_m - \Omega_n)} \int dx dy \left[\phi_m^*(x) \pi_n^*(y) [\phi(y, t), \pi(x, t)]^\dagger + \pi_m(x) \phi_n(y) [\phi(x, t), \pi(y, t)] \right] \\ &= ie^{it(\Omega_m - \Omega_n)} \int dx \left[\phi_m^*(x) \pi_n^*(x) - \pi_m(x) \phi_n(x) \right] \\ &= e^{it(\Omega_m - \Omega_n)} \langle \Psi_m | \Psi_n \rangle \\ &= \delta_{m,n} , \end{aligned}$$

which is what we wanted to prove. As Ambjørn and Wolfram we can then naturally define the vacuum state, $|0\rangle$, to have the property

$$a_n |0\rangle = b_n |0\rangle = 0 \quad \forall n . \quad (3.31)$$

¹¹Remember that $\Psi(x, t) = \begin{pmatrix} \phi(x, t) \\ (\pi(x, t))^* \end{pmatrix}$. Similarly, $\Psi_m^{(+)}(x) = \begin{pmatrix} \phi_m^{(+)}(x) \\ (\pi_m^{(+)}(x))^* \end{pmatrix}$

¹²The superscript (+) is omitted in the following. The commutation relations for b and b^\dagger are shown similarly by taking the inner product with $\Psi_m^{(-)}(x)$ and $\Psi_n^{(-)}(x)$.

3.5 Vacuum Polarization according to the Summation of Modes Method

Ambjørn and Wolfram define the charge density in the vacuum state as (cf. [1])

$$\begin{aligned}\rho(x) &= \langle 0 | \hat{\rho}(x) | 0 \rangle \\ \hat{\rho}(x) &= \frac{ie}{2} [\phi^\dagger(x) \pi^\dagger(x) + \pi^\dagger(x) \phi^\dagger(x) - \phi(x) \pi(x) - \pi(x) \phi(x)] .\end{aligned}\tag{3.32}$$

By using the commutation relations found above (3.27) one easily shows that

$$\langle 0 | a_m a_n^\dagger | 0 \rangle = \langle 0 | b_m b_n^\dagger | 0 \rangle = \delta_{m,n} .$$

The charge density in the vacuum state is then straightforwardly found to be:

$$\rho(x) = \frac{1}{2} \sum_n \left(2e(\Omega_n^{(+)} - eA_0) |\phi_n^{(+)}|^2 + 2e(\Omega_n^{(-)} - eA_0) |\phi_n^{(-)}|^2 \right) .\tag{3.33}$$

From equation (3.25) one may make the following identification

$$\pi_n^{(\pm)} = i(\Omega_n^{(\pm)} - eA_0)(\phi_n^{(\pm)})^* ,$$

and in analogy to the definition of the charge density in Lagrangian field theory (3.8), we may then define¹³

$$\rho_n^{(\pm)} := ie \left((\phi_n^{(\pm)})^* (\pi_n^{(\pm)})^* - \phi_n^{(\pm)} \pi_n^{(\pm)} \right) = 2e(\Omega_n^{(\pm)} - eA_0) |\phi_n^{(\pm)}|^2 .\tag{3.34}$$

The charge density according to the summation of modes method may then be written as (cf. [1])

$$\rho(x) = \frac{1}{2} \sum_n \left(\rho_n^{(+)}(x) + \rho_n^{(-)}(x) \right) .\tag{3.35}$$

This is the formula which Ambjørn and Wolfram used to evaluate the charge density. It can be seen (Equation (3.32)) that this method explicitly contains the product of the field ϕ with its conjugate momentum π at coinciding points. We will see later, that the Hadamard point-splitting procedure, which explicitly takes the singular part of such expressions into account, yields qualitatively different results for the charge density.

3.6 Vacuum Polarization from the Two-point Function

The charge density is from Lagrangian field theory given by (Eq. (3.8)):

$$\rho(x) = ie [\phi^*(x) (D_0 \phi(x)) - (D_0 \phi(x))^* \phi(x)]$$

In quantum field theory, the product of the fields is not well-defined at coinciding points, wherefore an alternative definition of the charge density should be used. Here we will use a method which we will refer to as the Hadamard point-splitting procedure.

¹³Ambjørn and Wolfram defined ρ_n to be the "charge density associated with each mode" (cf. [1]), which presumably means the same equation, but an exact definition was not given.

This method uses the two-point function to calculate the charge density. We will here define the two-point functions in the following way:

$$\begin{aligned}\omega_2\phi\phi^*(x,y) &:= \langle 0|\phi(x)\phi^*(y)|0\rangle = \sum_{n,+} \phi_n(x^1) (\phi_n(y^1))^* e^{-i\Omega_n(x_0-y_0)} \\ \omega_2\phi^*\phi(x,y) &:= \langle 0|\phi^*(y)\phi(x)|0\rangle = \sum_{n,-} \phi_n(x^1) (\phi_n(y^1))^* e^{-i\Omega_n(x_0-y_0)}\end{aligned}\tag{3.36}$$

In this procedure, one assumes that the two-point function has singularities of Hadamard form and subtracts these before the charge density is computed. For the Klein-Gordon field, we may explicitly define the charge density with this method as

$$\rho(x) = \lim_{y \rightarrow x} ie \left(D_0^x [\omega_2\phi^*\phi(x,y) - h^-] - (D_0^y)^* [\omega_2\phi\phi^*(x,y) - h^+] \right). \tag{3.37}$$

When calculating the charge density explicitly, one must therefore calculate the two-point functions and the parametrix h^+ and h^- separately and then subtract these. We will do this explicitly later.

Chapter 4

A Perturbative Solution to the Klein-Gordon Field on a Finite Interval

4.1 The Unperturbed State

We would now like to find a perturbative solution for small electric fields. For this reason the Hamiltonian is written as (see section 3.3)

$$H = H_0 + H_1$$

$$H_0 = i \begin{pmatrix} 0 & 1 \\ -(-D_i^2 + m^2) & 0 \end{pmatrix}, \quad H_1 = \begin{pmatrix} eA_0 & 0 \\ 0 & eA_0 \end{pmatrix}, \quad (4.1)$$

where H_1 is considered to be a small perturbation. In 1+1 dimensions the unperturbed problem reads:

$$i \begin{pmatrix} 0 & 1 \\ \partial_Z^2 - m^2 & 0 \end{pmatrix} \Psi_n = E_n \Psi_n \quad (4.2)$$

With the notation $\Psi_n = \begin{pmatrix} \phi_n \\ \pi_n^* \end{pmatrix}$ we get the following system of equations:

$$i\pi_n^* = \Omega_n \phi_n$$

$$i(\partial_Z^2 - m^2) \phi_n = \Omega_n \pi_n^* \quad (4.3)$$

It is however rather easily solved by putting the first equation into the second:

$$\partial_Z^2 \phi_n = -(\Omega_n^2 - m^2) \phi_n \quad (4.4)$$

For Dirichlet boundary conditions ($\phi(Z=0) = \phi(Z=a) = 0$), the solution is given by:

$$\Psi_n = C_n \begin{pmatrix} \sin(n\pi z) \\ -i \operatorname{sgn}(n) \sqrt{\left(\frac{n\pi}{a}\right)^2 + m^2} \sin(n\pi z) \end{pmatrix}, \quad n \in \mathbb{Z} \setminus \{0\}$$

$$\Omega_n = \operatorname{sgn}(n) \sqrt{\left(\frac{n\pi}{a}\right)^2 + m^2}, \quad n \in \mathbb{Z} \setminus \{0\} \quad (4.5)$$

where C_n is a normalization constant and $z = \frac{Z}{a}$. For Neumann boundary conditions ($\partial_Z \phi(Z=0) = \partial_Z \phi(Z=a) = 0$) one similarly obtains:

$$\begin{aligned}\Psi_n &= C_n \begin{pmatrix} \cos(n\pi z) \\ -i \operatorname{sgn}(n) \sqrt{\left(\frac{n\pi}{a}\right)^2 + m^2} \cos(n\pi z) \end{pmatrix}, \quad n \in \mathbb{Z} \setminus \{0\} \\ \Omega_n &= \operatorname{sgn}(n) \sqrt{\left(\frac{n\pi}{a}\right)^2 + m^2}, \quad n \in \mathbb{Z} \setminus \{0\} \\ \Psi_0^\pm &= C_0^\pm \begin{pmatrix} 1 \\ \mp im \end{pmatrix} \\ \Omega_0^\pm &= \pm m\end{aligned}\tag{4.6}$$

One should note that there are two zero modes, one with positive frequency and one with negative frequency.

We go on by calculating the normalization of the eigenvectors with respect to the indefinite inner product $\langle \cdot | \cdot \rangle$ (Eq. (3.22))

$$\langle \Psi_1 | \Psi_2 \rangle = i \int_0^a dZ (\phi_1^* \pi_2^* - \pi_1 \phi_2).$$

Choosing the normalization¹ +1 for the positive frequency modes and -1 for the negative frequency modes, one gets:

$$\begin{aligned}C_n &= \frac{1}{\sqrt[4]{(n\pi)^2 + (am)^2}} \\ C_0^\pm &= \frac{1}{\sqrt{2am}}\end{aligned}\tag{4.7}$$

Note that the value for C_n applies to both Dirichlet and Neumann boundary conditions. The normalized unperturbed eigenvectors are therefore given by:

$$\begin{aligned}\Psi_n &= \frac{1}{\sqrt[4]{(n\pi)^2 + (am)^2}} \begin{pmatrix} \sin(n\pi z) \\ -i \operatorname{sgn}(n) \sqrt{\left(\frac{n\pi}{a}\right)^2 + m^2} \sin(n\pi z) \end{pmatrix} \quad (\text{Dirichlet}) \\ \Psi_n &= \frac{1}{\sqrt[4]{(n\pi)^2 + (am)^2}} \begin{pmatrix} \cos(n\pi z) \\ -i \operatorname{sgn}(n) \sqrt{\left(\frac{n\pi}{a}\right)^2 + m^2} \cos(n\pi z) \end{pmatrix} \quad (\text{Neumann}) \\ \Psi_0^\pm &= \frac{1}{\sqrt{2am}} \begin{pmatrix} 1 \\ \mp im \end{pmatrix} \quad (\text{Neumann})\end{aligned}\tag{4.8}$$

The energies are for both cases given by $\Omega_n = \operatorname{sgn}(n) \sqrt{\left(\frac{n\pi}{a}\right)^2 + m^2}$. We further note that all the eigenvectors are not neutral, such that the eigenvectors of the massive Klein-Gordon equation with Dirichlet and Neumann boundary conditions can be used for the perturbation theory described in section 2.4.

¹This corresponds to choosing the normalization 1 with respect to the intrinsic (positive-definite) norms $|x|_\pm = \sqrt{\pm \langle x | x \rangle_\pm}$ on the subspaces K^\pm .

4.2 First Order Correction of the Energies

We are now in a position to start finding perturbative corrections to the modes for small electric fields. From section 2.4, the first order correction to the eigenvalues is given by equation (2.16)

$$\Omega_n^{(1)} = \text{sgn}(n) \left\langle \Psi_n^{(0)} \left| H_1 \Psi_n^{(0)} \right. \right\rangle .$$

Remember that we are using the normalization $f(n) = \text{sgn}(n)$. Using the results found above, we find for the Dirichlet case

$$\begin{aligned} \Omega_n^{(1)} &= \text{sgn}(n) i \int_0^a dZ \left(-i \frac{\text{sgn}(n)}{a} eA_0(z) \sin^2(n\pi z) - i \frac{\text{sgn}(n)}{a} eA_0(z) \sin^2(n\pi z) \right) \\ &= 2 \int_0^1 dz eA_0(z) \sin^2(n\pi z) . \end{aligned} \quad (4.9)$$

With $eA_0 = -\frac{\lambda}{a}(z - \frac{1}{2} + \alpha)$ one easily finds:

$$\Omega_n^{(1)} = -\frac{\lambda}{a} \alpha . \quad (4.10)$$

We thus see that the first order correction is proportional to the gauge parameter and that it is independent of n . Especially, it holds that $\omega_n = a\Omega_n + \lambda\alpha$ does not change to first order in λ . Hence

$$\omega_n = -\omega_{-n} , \quad (4.11)$$

to first order in λ . One easily finds that the exact same statements hold for the Neumann case.

4.3 Calculation of the Modes for Dirichlet Boundary Conditions

According to (2.19), the first order perturbative correction to the n th mode is given by:

$$\Psi_n^{(1)} = \sum_{k \in I \setminus \{n\}} \frac{1}{\langle \Psi_k^{(0)} | \Psi_k^{(0)} \rangle} \frac{\langle \Psi_k^{(0)} | H_1 \Psi_n^{(0)} \rangle}{\Omega_n^{(0)} - \Omega_k^{(0)}} \Psi_k^{(0)}$$

We will now calculate this explicitly. Recall that $H_1 = \begin{pmatrix} eA_0 & 0 \\ 0 & eA_0 \end{pmatrix}$ and that $eA_0(z) = -\frac{\lambda}{a}(z - \frac{1}{2} + \alpha)$. For the Dirichlet case, the first order correction is therefore given by²:

$$\begin{aligned} \phi_n^{(1)}(z) &= \sum_{k \in \mathbb{Z} \setminus \{0, n, -n\}} \text{sgn}(k) \frac{\text{sgn}(n) \sqrt{(am)^2 + (n\pi)^2} + \text{sgn}(k) \sqrt{(am)^2 + (k\pi)^2}}{\sqrt[4]{(am)^2 + (n\pi)^2} \sqrt[4]{(am)^2 + (k\pi)^2}} \\ &\quad \frac{\int_0^1 \sin(n\pi x) \sin(k\pi x) \left(-\lambda \left(x - \frac{1}{2} + \alpha \right) \right) \frac{\sin(k\pi z)}{\sqrt[4]{(am)^2 + (k\pi)^2}} dx}{\text{sgn}(n) \sqrt{(am)^2 + (n\pi)^2} - \text{sgn}(k) \sqrt{(am)^2 + (k\pi)^2}} \end{aligned}$$

²The term $k = -n$ was excluded, as it can be seen that it yields no contribution.

We now multiply and divide this expression by $\text{sgn}(n)\sqrt{(am)^2 + (n\pi)^2} + \text{sgn}(k)\sqrt{(am)^2 + (k\pi)^2}$. This yields:

$$\begin{aligned}\phi_n^{(1)}(z) = & -\frac{\lambda}{\sqrt[4]{(am)^2 + (n\pi)^2}} \sum_{k \in \mathbb{Z} \setminus \{0, n, -n\}} \text{sgn}(k) \\ & \times \frac{((am)^2 + (n\pi)^2) + 2 \text{sgn}(n) \text{sgn}(k) \sqrt{(am)^2 + (n\pi)^2} \sqrt{(am)^2 + (k\pi)^2} + ((am)^2 + (k\pi)^2)}{\sqrt{(am)^2 + (k\pi)^2}} \\ & \times \frac{\int_0^1 x \sin(n\pi x) \sin(k\pi x) \sin(k\pi z) dx}{(n\pi)^2 - (k\pi)^2}\end{aligned}$$

The first and third term will vanish³ as they are antisymmetric in k . Therefore:

$$\phi_n^{(1)}(z) = -\frac{\lambda \sqrt[4]{(am)^2 + (n\pi)^2}}{|n|\pi} \sum_{k \in \mathbb{N} \setminus \{n\}} \frac{4n\pi}{(n\pi)^2 - (k\pi)^2} \int_0^1 x \sin(n\pi x) \sin(k\pi x) \sin(k\pi z) dx \quad (4.12)$$

We note that if one were to exchange the order of summation and integration, the resulting sum would still be absolutely converging. According to the Fubini-Tonelli theorem, it is therefore allowed to interchange the order of summation and integration. To find the first order correction, it now remains to calculate the following expression:

$$-\int_0^1 \sum_{k \in \mathbb{N} \setminus \{n\}} \frac{4n\pi}{(n\pi)^2 - (k\pi)^2} x \sin(n\pi x) \sin(k\pi x) \sin(k\pi z) dx \quad (4.13)$$

As $\frac{4n}{n^2 - k^2} = \frac{2}{n - k} + \frac{2}{n + k}$ we may again write this as a sum over \mathbb{Z} :⁴

$$-\frac{1}{\pi} \int_0^1 \sum_{k \in \mathbb{Z} \setminus \{0, n, -n\}} \frac{2}{n - k} x \sin(n\pi x) \sin(k\pi x) \sin(k\pi z) dx \quad (4.14)$$

We proceed by rewriting this using the addition theorem of the cosine

$$\sin(k\pi x) \sin(k\pi z) = \frac{1}{2} (\cos(k\pi(x - z)) - \cos(k\pi(x + z))) , \quad (4.15)$$

and changing summation index to $j := n - k$, $j \in \mathbb{Z} \setminus \{0, n, 2n\}$. (4.13) becomes:

$$-\frac{1}{\pi} \int_0^1 \sum_{k \in \mathbb{Z} \setminus \{0, n, 2n\}} \frac{1}{j} x \sin(n\pi x) (\cos(k\pi(x - z)) - \cos(k\pi(x + z))) dx \quad (4.16)$$

We may now add and subtract the terms corresponding to $j = n$ and $j = 2n$ in order to sum over a symmetric region. The term $j = n$ yields no contribution, where as the

³Note that the sum is absolutely converging as $\int_0^1 x \cos(n\pi x) \cos(k\pi x) dx$ goes like $\frac{1}{k^2}$ for large k .

⁴One easily shows that the individual sums are converging using Dirichlet's test, which says that a real sequence, which is monotonically decreasing to 0, multiplied by a complex sequence, whose partial sum is bounded, converges (see e.g. [17]).

term $j = 2n$ yields the additional term

$$C = \frac{1}{n\pi} \int_0^1 x \sin^2(n\pi x) \sin(n\pi z) dx . \quad (4.17)$$

We may once again use the addition theorem of the cosine to get:

$$\begin{aligned} & \cos((n-j)\pi(x-z)) - \cos((n-j)\pi(x+z)) \\ &= \cos(n\pi(x-z)) \cos(j\pi(x-z)) + \sin(n\pi(x-z)) \sin(j\pi(x-z)) \\ & - \cos(n\pi(x+z)) \cos(j\pi(x+z)) - \sin(n\pi(x+z)) \sin(j\pi(x+z)) \end{aligned} \quad (4.18)$$

Note that the terms involving the cosine of j will vanish⁵, as they are antisymmetric in j . We now use the formula⁶ (cf. [18, p. 449])

$$\sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{\sin(j\pi y)}{j} = \begin{cases} \pi(1-y) & \text{if } 0 < y < 2 \\ \pi(-1-y) & \text{if } -2 < y < 0 \end{cases} . \quad (4.19)$$

We note that $x+z \in [0, 2]$. However, $x-z$ might be negative so we have to split the integral into two terms: $0 \leq x \leq z$ and $z \leq x \leq 1$. Equation (4.13) may therefore be written as:

$$\begin{aligned} C - \int_0^1 dx \, x \sin(n\pi x) & \left(-(x-z) \sin(n\pi(x-z)) - (1-(x+z)) \sin(n\pi(x+z)) \right) \\ & - \left(- \int_0^z x \sin(n\pi x) \sin(n\pi(x-z)) \, dx + \int_z^1 x \sin(n\pi x) \sin(n\pi(x-z)) \, dx \right) . \end{aligned} \quad (4.20)$$

Using the addition theorem of the sine function

$$\begin{aligned} \sin(n\pi(x-z)) - \sin(n\pi(x+z)) &= -2 \cos(n\pi x) \sin(n\pi z) \\ \sin(n\pi(x-z)) + \sin(n\pi(x+z)) &= 2 \sin(n\pi x) \cos(n\pi z) , \end{aligned} \quad (4.21)$$

we find that equation (4.13) may be written as:

$$\begin{aligned} C - 2 \int_0^1 x \sin(n\pi x) & (x \cos(n\pi x) \sin(n\pi z) + z \sin(n\pi x) \cos(n\pi z)) dx \\ & + 2 \left(\int_0^z x \sin^2(n\pi x) \cos(n\pi z) dx + \int_z^1 x \sin(n\pi x) \cos(n\pi x) \sin(n\pi z) dx \right) . \end{aligned} \quad (4.22)$$

We have now rewritten the expression such that it may easily (although tediously) be evaluated by integration by parts. Performing the integrals and simplifying using trigonometric identities, we finally obtain:

$$\begin{aligned} & - \int_0^1 \sum_{k \in \mathbb{N} \setminus \{n\}} \frac{4n\pi}{(n\pi)^2 - (k\pi)^2} x \sin(n\pi x) \sin(k\pi x) \sin(k\pi z) dx \\ &= \frac{1}{2n\pi} \left(\frac{1}{2} - z \right) \sin(n\pi z) - \frac{1}{2} z (1-z) \cos(n\pi z) \end{aligned} \quad (4.23)$$

⁵The convergence of the individual sums are still granted by Dirichlet's test.

⁶The sum is equal to 0 on the nullset $y = -2, 0, 2$.

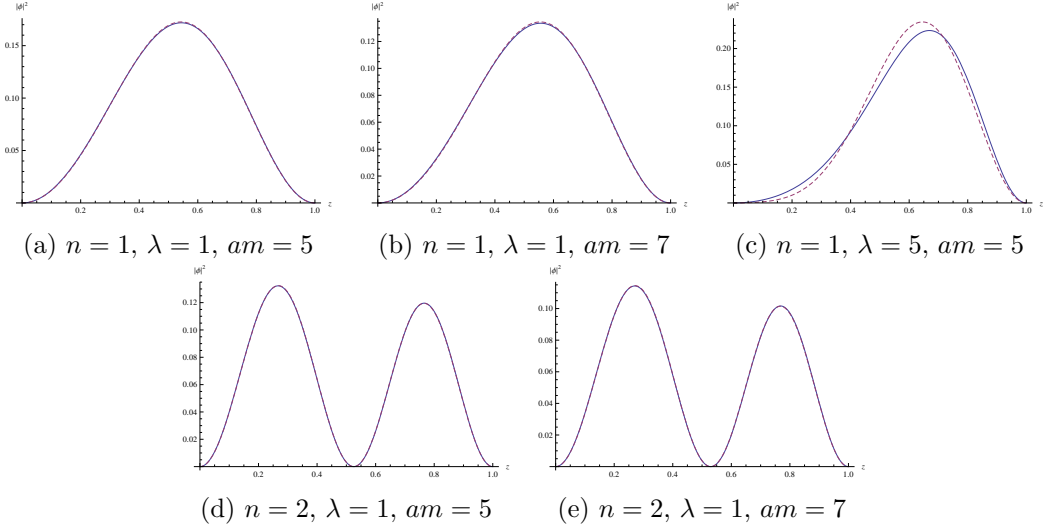


Figure 4.1: $|\phi_{n,pert}^{(1)}|^2$ (dashed, red line) and the numerical solution in terms of parabolic cylinder functions (blue line) are shown for different values of n , λ and am for Dirichlet boundary conditions.

The modes for Dirichlet boundary conditions are to first order in λ therefore given by:

$$\phi_{n,pert}^{(1)} = \frac{1}{\sqrt[4]{(am)^2 + (n\pi)^2}} \left\{ \sin(n\pi z) + \frac{\sqrt{(am)^2 + (n\pi)^2}}{|n|\pi} \left[\frac{\lambda}{2\pi n} \left(\frac{1}{2} - z \right) \sin(n\pi z) - \frac{\lambda}{2} z(1-z) \cos(n\pi z) \right] \right\} \quad (4.24)$$

This reduces to the modes presented by Ambjørn and Wolfram when setting $m = 0$ (cf. [1]). These modes (Eq. (4.24)) are compared to numerical calculations of the analytical solution (3.15) in terms of parabolic cylinder functions in figure 4.1 for different values of the parameters n , λ and am .

4.4 Calculation of the Modes for Neumann Boundary Conditions

For the Neumann case, one also has to take the contribution from the zero modes into account in (2.19). Completely analogous to the above calculation for the Dirichlet case, one finds that the contribution from the *non-zero* modes to the first order correction of the n^{th} mode ($n \neq 0$) is given by:

$$\phi_{n \neq 0}^{(1)}(z) = -\frac{\lambda \sqrt[4]{(am)^2 + (n\pi)^2}}{|n|\pi} \sum_{k \in \mathbb{N} \setminus \{n\}} \frac{4n\pi}{(n\pi)^2 - (k\pi)^2} \int_0^1 x \cos(n\pi x) \cos(k\pi x) \cos(k\pi z) dx \quad (4.25)$$

One can evaluate this analogously to the Dirichlet case by using the appropriate trigonometric identities.⁷ However, one needs to be careful when adding and sub-

⁷This basically correspond to changing some plus signs into minus signs and replacing some sine functions by cosines and vice versa.

tracting the terms corresponding to $j = n, 2n$ after equation (4.16), as the term $j = n$ now yields a non-zero contribution. Fortunately, this term exactly cancels the contribution from the zero modes. In this manner, one finds that the non-zero modes for the Neumann case are to first order in λ given by:

$$\begin{aligned} \phi_{n,\text{pert}}^{(1)} = \frac{1}{\sqrt[4]{(am)^2 + (n\pi)^2}} & \left\{ \cos(n\pi z) + \frac{\sqrt{(am)^2 + (n\pi)^2}}{|n|\pi} \left[\frac{\lambda}{2\pi n} \left(\frac{1}{2} - z \right) \cos(n\pi z) \right. \right. \\ & \left. \left. + \frac{\lambda}{2} \left(z(1-z) + \frac{1}{n^2\pi^2} \right) \sin(n\pi z) \right] \right\} \end{aligned} \quad (4.26)$$

One calculates the correction to the zero modes similarly, by putting $n = 0$ in (2.19), i.e.

$$\begin{aligned} \phi_{0,\pm}^{(1)} = \sum_{k \in \mathbb{Z}} \frac{\text{sgn}(k)}{\pm m - \text{sgn}(k) \sqrt{\left(\frac{k\pi}{a}\right)^2 + m^2}} & \int_0^1 dx \frac{-\lambda(x - \frac{1}{2} + \alpha)}{\sqrt[4]{(k\pi)^2 + (am)^2} \sqrt{2am}} \\ & \times \left(\pm m + \text{sgn}(k) \sqrt{\left(\frac{k\pi}{a}\right)^2 + m^2} \right) \cos(k\pi x) \frac{\cos(k\pi z)}{\sqrt[4]{(k\pi)^2 + (am)^2}}. \end{aligned} \quad (4.27)$$

By partial integration one easily finds that the integral vanishes for even k and that it is given by

$$\int_0^1 x \cos(k\pi x) dx = \frac{-2}{(k\pi)^2}, \quad (4.28)$$

for odd k . Multiplying and dividing by the conjugate of the denominator, i.e. $\pm m - \text{sgn}(k) \sqrt{\left(\frac{k\pi}{a}\right)^2 + m^2}$, now yields:

$$\phi_{0,\pm}^{(1)} = \frac{-\lambda}{\sqrt{2am}} \sum_{k,\text{odd}} 2 \text{sgn}(k) \frac{(am)^2 \pm 2am \text{sgn}(k) \sqrt{(k\pi)^2 + (am)^2} + (k\pi)^2 + (am)^2}{\sqrt{(k\pi)^2 + (am)^2} k^4 \pi^4} \quad (4.29)$$

One then easily notices, that each term is absolutely converging, as the asymptotic behaviour of the slowest converging term goes like $k^{-5/2}$. We further see, that only one term, namely the one containing $\pm 2am \text{sgn}(k) \sqrt{(k\pi)^2 + (am)^2}$ is even in k and thus, which is ensured by the convergence of the individual terms, only this term gives a non-zero contribution. We are therefore left with:

$$\phi_{0,\pm}^{(1)} = \mp 4\lambda \sqrt{2am} \sum_{k>0,\text{odd}} \frac{\cos(k\pi z)}{k^4 \pi^4} \quad (4.30)$$

We now take advantage of the absolute convergence and rewrite the sum of odd natural numbers into a sum of all natural numbers minus the even ones. Formally, we perform the following rewriting:

$$\sum_{k>0,\text{odd}} f(k) \rightarrow \sum_{k=1}^{\infty} f(k) - \sum_{k=1}^{\infty} f(2k) \quad (4.31)$$

Thus, the first order correction to the zero modes can be written as:

$$\phi_{0,\pm}^{(1)} = \mp 4\lambda\sqrt{2am} \left(\sum_{k>0,\text{odd}} \frac{\cos(k\pi z)}{k^4\pi^4} - \frac{1}{16} \sum_{k>0,\text{odd}} \frac{\cos(2k\pi z)}{k^4\pi^4} \right) \quad (4.32)$$

With the knowledge of the sum (cf. [18, p. 449])

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^4} = \frac{1}{90}\pi^4 - \frac{1}{12}\pi^2 x^2 + \frac{1}{12}\pi x^3 - \frac{1}{48}x^4, \quad (4.33)$$

the calculation is now easy, and one obtains:

$$\phi_{0,\pm}^{(1)} = \mp \lambda\sqrt{2am} \left(\frac{1}{24} - \frac{1}{4}z^2 + \frac{1}{6}z^3 \right) \quad (4.34)$$

Summarizing, the modes for the Neumann case are to first order in λ given by:

$$\begin{aligned} \phi_{n,\text{pert}}^{(1)} &= \frac{1}{\sqrt[4]{(am)^2 + (n\pi)^2}} \left\{ \cos(n\pi z) + \frac{\sqrt{(am)^2 + (n\pi)^2}}{|n|\pi} \left[\frac{\lambda}{2\pi n} \left(\frac{1}{2} - z \right) \cos(n\pi z) \right. \right. \\ &\quad \left. \left. + \frac{\lambda}{2} \left(z(1-z) + \frac{1}{n^2\pi^2} \right) \sin(n\pi z) \right] \right\} \\ \phi_{0,\text{pert}}^{\pm,(1)} &= \frac{1}{\sqrt{2am}} \mp \lambda\sqrt{2am} \left[\frac{1}{24} - \frac{1}{4}z^2 + \frac{1}{6}z^3 \right] \end{aligned} \quad (4.35)$$

These modes are compared to numerical calculations of the analytical solution (3.15) in terms of parabolic cylinder functions in figure 4.2.

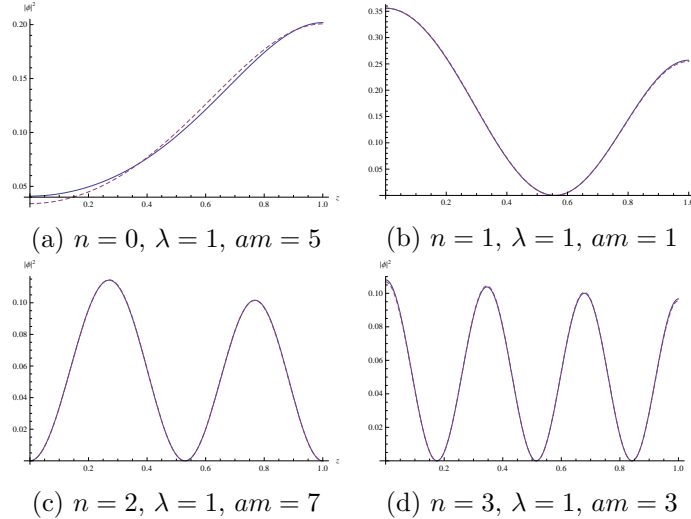


Figure 4.2: $|\phi_{n,\text{pert}}^{(1)}|^2$ (dashed, red line) and the numerical solution in terms of parabolic cylinder functions (blue line) are shown for different values of n , λ and am for Neumann boundary conditions.

Chapter 5

The Charge Density

5.1 The Special Case $m = 0$

For the special case $m = 0$ the charge density may be calculated explicitly for Dirichlet boundary conditions in first order perturbation theory. We will now do this using the summation of modes method and the Hadamard point-splitting procedure.

5.1.1 The Summation of Modes Method

For $m = 0$ the modes are for Dirichlet boundary conditions given by (Eq. (4.24)):

$$\phi_{n,pert}^{(1)} = \frac{1}{\sqrt{|n|\pi}} \left[\sin(n\pi z) \left(1 + \frac{\lambda}{2n\pi} \left(\frac{1}{2} - z \right) \right) - \frac{\lambda}{2} z(1-z) \cos(n\pi z) \right] . \quad (5.1)$$

According to the summation of modes method, the charge density is now calculated to first order in λ by using the formalism developed in section 3.4. We first calculate the charge density of each mode (Equation (3.34)) $\rho_n = 2e(\Omega_n - eA_0)|\phi_n|^2$. For $A_0(z) = -\frac{\lambda}{a} \left(z - \frac{1}{2} + \alpha \right)$ one finds:

$$\rho_n(z) = 2\frac{e}{a} \left(\omega_n + \lambda \left(z - \frac{1}{2} \right) \right) |\phi_n(z)|^2 . \quad (5.2)$$

Using the above expression for the modes and $\omega_{-n} = -\omega_n = n\pi$ (Equation (4.11)), it follows that

$$\begin{aligned} \rho_n^I &= \frac{1}{2} (\rho_n + \rho_{-n}) \\ &= -\lambda \frac{e}{a} z(1-z) \sin(2n\pi z) . \end{aligned} \quad (5.3)$$

Unfortunately, $\sum_{n=1}^{\infty} \rho_n^I$ is not converging for most z , which is easily seen by noting that $\sin(2n\pi z)$ is not converging to zero, which is a necessary condition for the convergence of an infinite series. However, one can still make sense of this expression. For instance,

the sum is Abel summable¹, as

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2i} \sum_{n=1}^{\infty} \left[e^{i2n\pi(z+i\varepsilon)} - e^{-i2n\pi(z+i\varepsilon)} \right] &= \lim_{\varepsilon \rightarrow 0^+} \frac{\sin(2\pi(z+i\varepsilon))}{2 - 2\cos(2\pi(z+i\varepsilon))} \\ &= \frac{1 \cos(\pi z)}{2 \sin(\pi z)} . \end{aligned}$$

Hence, ρ_n^I Abel converges to

$$\rho^I(z) = -\frac{1}{2} \frac{e}{a} \lambda z(1-z) \cot(\pi z) . \quad (5.4)$$

This is the charge density given by Ambjørn and Wolfram [1]. One can also show that $\sum_{n=1}^{\infty} \rho_n^I$ converges to $\rho^I(z)$ in a distributional sense, by showing that for any smooth test function $f(z)$

$$\sum_{n=1}^{\infty} \int_0^1 \rho_n^I(z) f(z) dz = \int_0^1 \rho^I(z) f(z) dz . \quad (5.5)$$

This calculation is showed in appendix B. ρ^I is plotted in figure 5.1. One especially notes that the charge density does not vanish on the boundary according to this method.

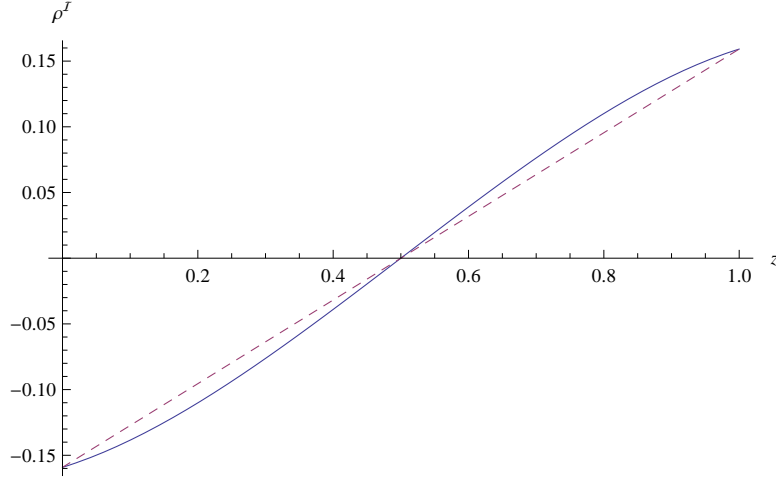


Figure 5.1: Plot of $\rho^I(z)$ (full line) in units of $\frac{e}{a} \lambda$. The line $-\frac{1}{2\pi} + \frac{z}{\pi}$ (dashed line) is given as comparison.

5.1.2 Explicit Calculation of the Two-point Function

We will here calculate the charge density for the Dirichlet boundary conditions using the Hadamard point-splitting procedure, by first computing the functions $(D_0^y)^* \omega_2 \phi \phi^*(x, y)$

¹A sum $\sum a_n$ is Abel summable, if $\sum a_n z^n$ converges for all $|z| < 1$ and if $\lim_{z \rightarrow 1^-} \sum a_n z^n$ exists, cf. [17]

and $D_0^x \omega_2 \phi^* \phi(x, y)$. We should therefore evaluate:²

$$\begin{aligned} (D_0^y)^* \omega_2 \phi^* \phi(x, y) &= i \sum_{n,+} \phi_n(x^1) (\Omega_n - eA_0(y^1)) (\phi_n(y^1))^* e^{-i(x^0-y^0)\Omega_n} \\ D_0^x \omega_2 \phi^* \phi(x, y) &= -i \sum_{n,-} \phi_n(x^1) (\Omega_n - eA_0(x^1)) (\phi_n(y^1))^* e^{-i(x^0-y^0)\Omega_n} \end{aligned} \quad (5.6)$$

For $A_0(z^1) = -\frac{\lambda}{ea} (z^1 - \frac{1}{2} + \alpha)$, $\omega_n = a\Omega_n + \alpha\lambda$, we get:

$$\begin{aligned} (D_0^y)^* \omega_2 \phi^* \phi(x, y) &= \frac{i}{a} \sum_{n,+} \phi_n(x^1) \left(\omega_n + \lambda \left(y^1 - \frac{1}{2} \right) \right) (\phi_n(y^1))^* e^{-i \frac{x^0-y^0}{a} (\omega_n - \alpha\lambda)} \\ D_0^x \omega_2 \phi^* \phi(x, y) &= -\frac{i}{a} \sum_{n,-} \phi_n(x^1) \left(\omega_n + \lambda \left(x^1 - \frac{1}{2} \right) \right) (\phi_n(y^1))^* e^{-i \frac{x^0-y^0}{a} (\omega_n - \alpha\lambda)} \end{aligned} \quad (5.7)$$

To first order in λ :³

$$\begin{aligned} \phi_n(z) &= \frac{1}{\sqrt{|n|\pi}} \left[\sin(n\pi z) \left(1 + \frac{\lambda}{2n\pi} \left(\frac{1}{2} - z \right) \right) - \frac{\lambda}{2} z(1-z) \cos(n\pi z) \right] \\ \omega_n &= n\pi \end{aligned} \quad (5.8)$$

Therefore, to first order in λ :

$$\begin{aligned} &\phi_n(x^1) \left(\omega_n + \lambda \left(y^1 - \frac{1}{2} \right) \right) (\phi_n(y^1))^* \\ &= \text{sgn}(n) \sin(n\pi x^1) \left[\sin(n\pi y^1) \left(1 + \frac{\lambda}{2n\pi} \left(\frac{1}{2} - y^1 \right) \right) - \frac{\lambda}{2} y^1(1-y^1) \cos(n\pi y^1) \right] \\ &\quad + \text{sgn}(n) \lambda \left[\frac{1}{2n\pi} \sin(n\pi x^1) \left(\frac{1}{2} - x^1 \right) - \frac{1}{2} x^1(1-x^1) \cos(n\pi x^1) \right] \sin(n\pi y^1) \\ &\quad + \frac{\lambda}{|n|\pi} \sin(n\pi x^1) \left(y^1 - \frac{1}{2} \right) \sin(n\pi y^1) \end{aligned}$$

The term $\phi_n(x^1) (\omega_n + \lambda (x^1 - \frac{1}{2})) (\phi_n(y^1))^*$ is obtained by performing the transformation $x^1 \leftrightarrow y^1$ in the above expression. We must therefore calculate sums of the form $\sin(n\pi x^1) \sin(n\pi y^1)$, $\sin(n\pi x^1) \cos(n\pi y^1)$ and $\frac{\sin(n\pi x^1) \sin(n\pi y^1)}{n}$, each multiplied by a factor $e^{-i \frac{n\pi}{a} (x^0 - y^0)}$. To evaluate these sums we will use the $i\varepsilon$ -prescription, which can be thought of as adding a small imaginary part, $i\varepsilon$, to one of the time-components.

We introduce the notations $x = \pi x^1$, $y = \pi y^1$ and $t = \frac{\pi(x^0 - y^0)}{a}$. The relevant sums

² x^1 and y^1 are here reduced spatial variables, that is, divided by a .

³This is found by setting $m = 0$ in 4.24.

expanded to first order in $(x - y - t + i\varepsilon)$ are given by⁴

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sin(nx) \sin(ny) e^{-in(t-i\varepsilon)} \\
&= \frac{-1}{4} \left(\frac{1}{1 - e^{i(x+y-t+i\varepsilon)}} - \frac{i}{x - y - t + i\varepsilon} - \frac{i}{-x + y - t + i\varepsilon} \right. \\
&\quad \left. + \frac{1}{1 - e^{i(-x-y-t+i\varepsilon)}} - 1 - \frac{i}{6}(t - i\varepsilon) \right), \\
& \sum_{n=1}^{\infty} \sin(nx) \cos(ny) e^{-in(t-i\varepsilon)} \\
&= \frac{-i}{4} \left(\frac{1}{1 - e^{i(x+y-t+i\varepsilon)}} + \frac{i}{x - y - t + i\varepsilon} - \frac{i}{-x + y - t + i\varepsilon} \right. \\
&\quad \left. - \frac{1}{1 - e^{i(-x-y-t+i\varepsilon)}} - \frac{i}{6}(x - y) \right), \\
& \sum_{n=1}^{\infty} \frac{\sin(nx) \sin(ny)}{n} e^{-in(t-i\varepsilon)} \\
&= \frac{1}{4} \left(\log \left(1 - e^{i(x+y-t+i\varepsilon)} \right) - \log(x - y - t + i\varepsilon) \right. \\
&\quad \left. - \log(-x + y - t + i\varepsilon) + \log \left(1 - e^{i(-x-y-t+i\varepsilon)} \right) + i\pi + i(t - i\varepsilon) \right).
\end{aligned}$$

⁴These are easily found by writing the trigonometric functions in terms of exponentials and then using the geometric series or the Taylor expansion of the logarithm. One then expands the results to linear order in $(x - y - t + i\varepsilon)$.

Using the above expressions we now obtain:

$$\begin{aligned}
(D_0^y)^* \omega_2 \phi \phi^*(x, y) = & \pm \frac{i}{a} e^{i \frac{x^0 - y^0}{a} \lambda \alpha} \left[\right. \\
& \pm \frac{-1}{4} \left(\frac{1}{1 - e^{i(x+y-t+i\varepsilon)}} - \frac{i}{x-y-t+i\varepsilon} - \frac{i}{-x+y-t+i\varepsilon} \right. \\
& + \left. \frac{1}{1 - e^{i(-x-y-t+i\varepsilon)}} - 1 - \frac{i}{6}(t-i\varepsilon) \right) \\
& \mp \frac{\lambda}{2\pi} \left(\frac{1}{2} - y^1 \right) \frac{1}{4} \left(\log \left(1 - e^{i(x+y-t+i\varepsilon)} \right) - \log(x-y-t+i\varepsilon) \right. \\
& - \log(-x+y-t+i\varepsilon) + \log \left(1 - e^{i(-x-y-t+i\varepsilon)} \right) + i\pi + i(t-i\varepsilon) \Big) \\
& - \frac{\lambda}{2} y^1 (1 - y^1) \frac{-i}{4} \left(\frac{1}{1 - e^{i(x+y-t+i\varepsilon)}} + \frac{i}{x-y-t+i\varepsilon} - \frac{i}{-x+y-t+i\varepsilon} \right. \\
& - \left. \frac{1}{1 - e^{i(-x-y-t+i\varepsilon)}} - \frac{i}{6}(x-y) \right) \\
& \pm \frac{\lambda}{2\pi} \left(\frac{1}{2} - x^1 \right) \frac{1}{4} \left(\log \left(1 - e^{i(x+y-t+i\varepsilon)} \right) - \log(x-y-t+i\varepsilon) \right. \\
& - \log(-x+y-t+i\varepsilon) + \log \left(1 - e^{i(-x-y-t+i\varepsilon)} \right) + i\pi + i(t-i\varepsilon) \Big) \\
& - \frac{\lambda}{2} x^1 (1 - x^1) \frac{-i}{4} \left(\frac{1}{1 - e^{i(x+y-t+i\varepsilon)}} + \frac{i}{-x+y-t+i\varepsilon} - \frac{i}{x-y-t+i\varepsilon} \right. \\
& - \left. \frac{1}{1 - e^{i(-x-y-t+i\varepsilon)}} + \frac{i}{6}(x-y) \right) \Big] \\
\end{aligned} \tag{5.9}$$

Here, the upper sign should be used for $(D_0^y)^* \omega_2 \phi \phi^*(x, y)$ and the lower sign for $D_0^x \omega_2 \phi^* \phi(x, y)$. Furthermore, one should do the replacement $t \rightarrow -t$ in the expression for $D_0^x \omega_2 \phi^* \phi(x, y)$. We further remind ourselves that with the above conventions $x = \pi x^1$, $y = \pi y^1$ and $t = \pi \frac{x^0 - y^0}{a}$, where x^1 and y^1 are the spatial variables divided by the interval length a . Now that we have managed to calculate the two-point function explicitly to first order in λ , we can use it to calculate the charge density.

5.1.3 The Hadamard Point-splitting Procedure

We will now evaluate the charge density according to (3.37) using the two-point functions calculated in the previous section. Using (2.26), we find that the parallel transport is in our case given by:

$$V_0(x, y) = e^{i\lambda \frac{x^0 - y^0}{a} \left(\frac{x^1 + y^1}{2} - \frac{1}{2} + \alpha \right)} \tag{5.10}$$

The singular parts of $D_0^x \omega_2 \phi^* \phi(x, y)$ and $(D_0^y)^* \omega_2 \phi \phi^*(x, y)$ can now be found by calculating $D_0^x h^-(x, y)$ and $(D_0^y)^* h^+(x, y)$. Using (2.27) and (2.28) one finds

$$\begin{aligned}
(D_0^y)^* h^+(x, y) &= \frac{-1}{4\pi} e^{i\lambda \frac{x^0 - y^0}{a} \left(\frac{x^1 + y^1}{2} - \frac{1}{2} + \alpha \right)} \frac{2(x^0 - y^0 - i\varepsilon)}{(x^1 - y^1)^2 + i\varepsilon(x^0 - y^0) - (x^0 - y^0)^2} \\
D_0^x h^-(x, y) &= \frac{1}{4\pi} e^{i\lambda \frac{x^0 - y^0}{a} \left(\frac{x^1 + y^1}{2} - \frac{1}{2} + \alpha \right)} \frac{2(x^0 - y^0 + i\varepsilon)}{(x^1 - y^1)^2 - i\varepsilon(x^0 - y^0) - (x^0 - y^0)^2}, \\
\end{aligned} \tag{5.11}$$

plus terms that vanish in the coinciding-point limit. If one takes the coinciding-point limit from the time direction, singularities in $(D_0^y)^* \omega_2 \phi \phi^*(x, y)$ occur in the first term in (5.9). However, one can see that $(D_0^y)^* h^+(x, y)$ cancels this singularity:⁵

$$\begin{aligned} & \lim_{y \rightarrow x} \frac{i}{a} e^{i \frac{x^0 - y^0}{a} \lambda \alpha} \frac{-1}{4} \left[\frac{-i}{x - y - t + i\varepsilon} - \frac{i}{-x + y - t + i\varepsilon} \right] - (D_0^y)^* h^+(x, y) \\ &= -\frac{i\lambda}{2\pi a} \left(x^1 - \frac{1}{2} \right) \end{aligned}$$

For $D_0^x \omega_2 \phi^* \phi(x, y)$, one similarly finds

$$\begin{aligned} & \lim_{y \rightarrow x} \frac{i}{a} e^{i \frac{x^0 - y^0}{a} \lambda \alpha} \frac{1}{4} \left[\frac{-i}{x - y + t + i\varepsilon} - \frac{i}{-x + y + t + i\varepsilon} \right] - D_0^x h^-(x, y) \\ &= \frac{i\lambda}{2\pi a} \left(x^1 - \frac{1}{2} \right) \end{aligned}$$

The charge density can now be calculated to first order in λ according to (3.37) by taking the limit from the time direction:⁶

$$\begin{aligned} \rho(x) &= \lim_{t \rightarrow 0} \lim_{\varepsilon \rightarrow 0^+} i e \left[\frac{i\lambda}{2\pi a} \left(x^1 - \frac{1}{2} \right) \right. \\ &\quad + \frac{\lambda}{4a} x^1 (1 - x^1) \left(\frac{1}{1 - e^{i(2x+t+i\varepsilon)}} - \frac{1}{1 - e^{i(-2x+t+\varepsilon)}} \right) \\ &\quad + \frac{i\lambda}{2\pi a} \left(x^1 - \frac{1}{2} \right) \\ &\quad \left. + \frac{\lambda}{4a} x^1 (1 - x^1) \left(\frac{1}{1 - e^{i(2x-t+i\varepsilon)}} - \frac{1}{1 - e^{i(-2x-t+\varepsilon)}} \right) \right] \quad (5.12) \\ &= 2ie \left[\frac{i\lambda}{2\pi a} \left(x^1 - \frac{1}{2} \right) + \frac{\lambda}{4a} x^1 (1 - x^1) \left(\frac{1}{1 - e^{2ix}} - \frac{1}{1 - e^{-2ix}} \right) \right] \\ &= -\frac{e}{a} \frac{\lambda}{\pi} \left(x^1 - \frac{1}{2} \right) - \frac{1}{2} \frac{e}{a} \lambda x^1 (1 - x^1) \cot(\pi x^1) \end{aligned}$$

This result is shown in figure 5.2. The result differs from the one obtained by the summation of modes method in that there is an extra term, which compensates the other term near the boundary of the interval and results in a vanishing charge density on the boundary of the interval. This result is therefore qualitatively different from the result presented by Ambjørn and Wolfram (cf. [1]), where the charge density takes on the values $\pm \frac{e\lambda}{2\pi a}$ on the boundary. It is however in some sense reasonable as the modes all vanish on the boundary of the interval due to the Dirichlet boundary conditions. In [19] a similar result was obtained: They calculated the vacuum expectation value of the charge density due to a constant gauge field in flat spacetimes with toroidal topology and found that it vanishes on the boundary for Dirichlet boundary conditions.

The results obtained above show explicitly that the summation of modes method and the Hadamard point-splitting procedure yield different results when applied to the

⁵To match the conventions used in h^+ and h^- one needs to do the replacement $\varepsilon \rightarrow \frac{\pi\varepsilon}{2a}$ in the expressions for the two-point functions. Also recall that $x = \pi x^1$ and $t = \frac{\pi(x^0 - y^0)}{a}$.

⁶One may also calculate the charge density with a coinciding-point limit from the space direction. One then proceeds analogously and the same result is obtained.

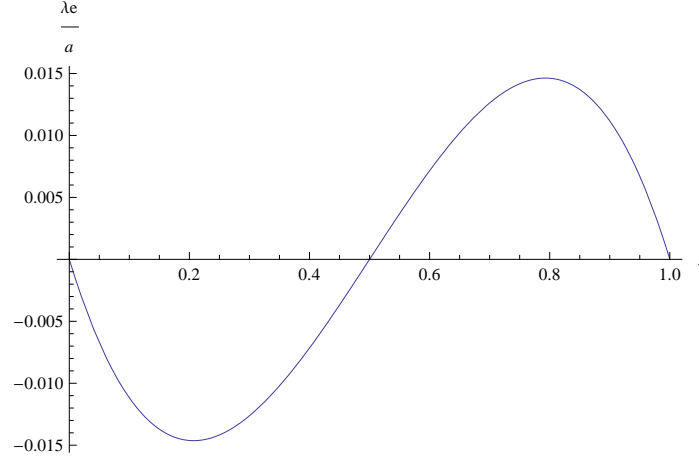


Figure 5.2: The vacuum polarization obtained by the Hadamard point-splitting procedure for the massless Klein-Gordon field on a finite interval in $1+1$ dimensions with Dirichlet boundary conditions.

Klein-Gordon field in a static external electric field, similar to what was found for the Dirac field in [2].

5.2 The Charge Density for $m \neq 0$

The charge density per mode for the massive case can be found by using (5.2). For Dirichlet boundary conditions, one finds, by using the modes found before (4.24), that the induced charge density for the n^{th} mode (Equation (5.3)) is given by:

$$\begin{aligned} \rho_n^I = & -\frac{\lambda e}{a} \left[\frac{\sqrt{(am)^2 + (n\pi)^2}}{n\pi} z(1-z) \sin(2n\pi z) \right. \\ & \left. + 2\left(z - \frac{1}{2}\right) \sin^2(n\pi z) \left(\frac{\sqrt{(am)^2 + (n\pi)^2}}{(n\pi)^2} - \frac{1}{\sqrt{(am)^2 + (n\pi)^2}} \right) \right] \end{aligned}$$

Using the modes found for Neumann boundary conditions (4.35), one similarly finds for the Neumann case:

$$\begin{aligned} \rho_n^I = & \frac{\lambda e}{a} \left[\frac{\sqrt{(am)^2 + (n\pi)^2}}{n\pi} \left(z(1-z) + \frac{1}{(n\pi)^2} \right) \sin(2n\pi z) \right. \\ & \left. - 2 \left(z - \frac{1}{2} \right) \cos^2(n\pi z) \left(\frac{\sqrt{(am)^2 + (n\pi)^2}}{(n\pi)^2} - \frac{1}{\sqrt{(am)^2 + (n\pi)^2}} \right) \right] \\ \rho_0^I = & \frac{\lambda e}{a} \left(\frac{(z - \frac{1}{2})}{am} - am \left[\frac{1}{6} - z^2 + \frac{2}{3}z^3 \right] \right) \end{aligned}$$

The charge density is now, according to the summation of modes method, given as $\rho = \sum_n \rho_n^I$. Unfortunately, we needed to evaluate these expressions numerically.

The above calculations suggest that, in the limit $m \rightarrow 0$, the charge density shows an extremely strong screening behaviour and that it is definitely not anti-screening,

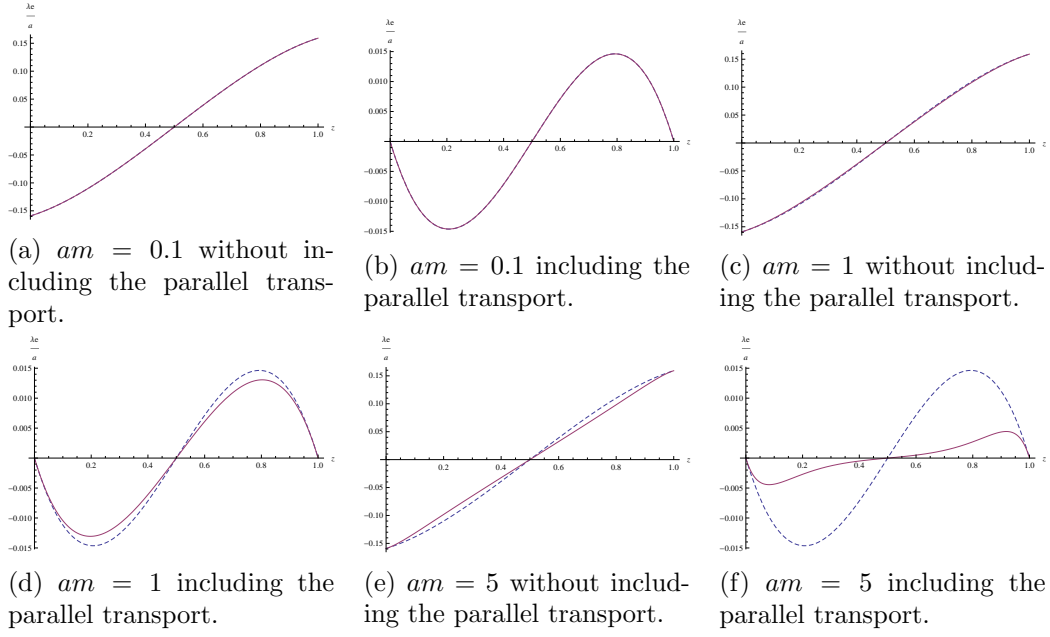


Figure 5.3: Numerical calculations of the vacuum polarization at different values of am for Dirichlet boundary conditions are shown. The dashed line shows the charge density for the massless case.

as suggested by Ambjørn and Wolfram within the external field approximation (cf. [1]). The reason that they got that result is of course not due to the fact that they used the summation of modes method (up to this point, we did as well), but rather that they neglected the contribution from the zero modes. As we can see explicitly by inspecting our results, the zero mode is actually dominating the charge density for small masses. However, it seems like e.g. back-reaction effects need to be included in order to understand the behaviour for small masses in the Neumann case properly, as one might physically not expect the vacuum polarization to grow indefinitely in the limit $m \rightarrow 0$. For instance, one would intuitively not expect the screening field to be larger than the external field.

The charge density can now be calculated numerically by summing the above expressions and including the term coming from the parallel transport. The results are presented in figure 5.3 and 5.4.

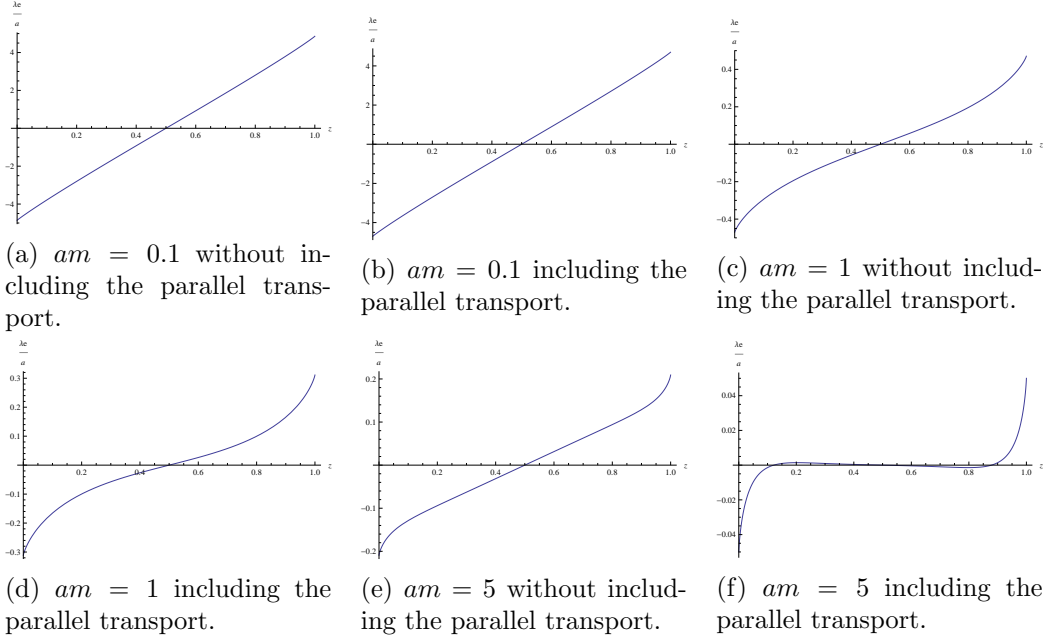


Figure 5.4: Numerical calculations of the vacuum polarization at different values of am for Neumann boundary conditions are shown.

We can further try to find closed form expressions for $am \ll \pi$ by Taylor expanding the above formulae. Doing this and performing the sum yields for the Dirichlet case:

$$\begin{aligned} \rho(z) = & -\lambda \frac{e}{2a} \left[z(1-z) \cot(\pi z) + \frac{(am)^2}{2i\pi^2} z(1-z) (\text{Li}_2(e^{2i\pi z}) - \text{Li}_2(e^{-2i\pi z})) \right. \\ & \left. + 2 \frac{(am)^2}{\pi^3} \left(z - \frac{1}{2} \right) \left(\zeta(3) - \frac{1}{2} (\text{Li}_3(e^{2i\pi z}) + \text{Li}_3(e^{-2i\pi z})) \right) \right] + O\left(\left(\frac{am}{\pi}\right)^4\right) \end{aligned} \quad (5.13)$$

Li_2 is the polylogarithm⁷ and ζ is the Riemann-zeta function.

For the Neumann case, one similarly gets

$$\begin{aligned} \rho(z) = & \lambda \frac{e}{2a} \left[\frac{2}{am} \left(z - \frac{1}{2} \right) + z(1-z) \cot(\pi z) \right. \\ & + \frac{1}{i\pi^2} (\text{Li}_2(e^{2i\pi z}) - \text{Li}_2(e^{-2i\pi z})) - 2am \left[\frac{1}{6} - z^2 + \frac{2}{3} z^3 \right] \\ & + \frac{(am)^2}{2i\pi^2} \left(z(1-z) (\text{Li}_2(e^{2i\pi z}) - \text{Li}_2(e^{-2i\pi z})) + \frac{1}{\pi^2} (\text{Li}_4(e^{2i\pi z}) - \text{Li}_4(e^{-2i\pi z})) \right) \\ & \left. - 2 \frac{(am)^2}{\pi^3} \left(z - \frac{1}{2} \right) \left(\zeta(3) + \frac{1}{2} (\text{Li}_3(e^{2i\pi z}) + \text{Li}_3(e^{-2i\pi z})) \right) \right] + O\left(\left(\frac{am}{\pi}\right)^4\right). \end{aligned} \quad (5.14)$$

⁷The polylogarithm is defined by $\text{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$ for $|z| < 1$. See [20] for properties of such functions.

Chapter 6

Conclusions and Outlook

In this work the massive Klein-Gordon field on a finite interval in $1 + 1$ dimensions subject to a static, external electric field was treated. The modes were calculated explicitly for Dirichlet and Neumann boundary conditions up to first order in the field. The analytic solution in terms of parabolic cylinder functions was also presented and a small discrepancy to the solution presented by Ambjørn and Wolfram in [1] in the massless limit was found. As a special case, we considered the massless case under Dirichlet boundary conditions, for which the charge density may be calculated explicitly with the summation of modes method, as well as with the Hadamard point-splitting procedure. When using the summation of modes method, the same result as those of Ambjørn and Wolfram were found, whereas the Hadamard point-splitting procedure yielded an extra term, which caused the charge density to vanish on the boundary of the interval. The two methods thus yielded qualitatively different results, as also found in [2] for the Dirac field.

The charge density per mode was further found explicitly for the massive case for both Dirichlet and Neumann boundary conditions, but to calculate the full charge density a numerical computation was needed. When the parallel transport was included one here also found a vanishing charge density on the boundary for Dirichlet boundary conditions. However, for Neumann boundary conditions, the charge density was found not to vanish on the boundary in general and its value was found to depend on the product of the interval length and the mass am . It was also found - even when the parallel transport was not included - that the field also for Neumann boundary conditions screens the external field, as one would physically expect. In the limit $m \rightarrow 0$ the screening behaviour was found to be especially strong, in contrary to the anti-screening behaviour claimed by Ambjørn and Wolfram in the massless case. The reason for this qualitative discrepancy was found to be due to that they neglected the contribution from the zero modes to the charge density, which was seen to be especially important in the limit of low masses.

It would be interesting to explore non-perturbative effects, by e.g. taking a more numerical approach to the problem. Then, one could also investigate whether the behaviour of the vacuum polarization far away from the boundary is depending on the boundary conditions in the limit of large intervals $a \rightarrow \infty$, or not. One would physically expect that the boundary conditions do not matter far away from the boundary for large intervals. One could further try to generalize the treatment in this work to other boundary conditions and/or higher dimensions. The field was here taken to be an external field, but one may try to leave this approximation by taking back-reaction

effects into account. One might for instance expect the vacuum polarization to not grow indefinitely for Neumann boundary conditions in the limit of low masses. One could further consider an electric field varying in space.

Appendix A

Propagators of the Klein-Gordon Field

We will now show that the retarded propagator of the Klein-Gordon field in a static electric field ($\partial_t A_0 = 0$) is given by:

$$\Delta^- = i\theta(x^0 - y^0) \langle 0 | [\phi(x), \phi^*(y)] | 0 \rangle \quad (\text{A.1})$$

A similar calculation for the free Klein-Gordon field is given in [3]. As such a calculation is usually not given when an external field is present, the calculations will be presented here. One finds:

$$\begin{aligned} \frac{1}{i} (D_\mu^2 + m^2) \Delta^- &= (\partial_t^2 \theta(x^0 - y^0)) \langle 0 | [\phi(x), \phi^*(y)] | 0 \rangle + 2 (\partial_t \theta(x^0 - y^0)) \langle 0 | [\partial_t \phi(x), \phi^*(y)] | 0 \rangle \\ &\quad + 2ieA_0 (\partial_t \theta(x^0 - y^0)) \langle 0 | [\phi(x), \phi^*(y)] | 0 \rangle \\ &\quad + \theta(x^0 - y^0) (D_\mu^2 + m^2) \langle 0 | [\phi(x), \phi^*(y)] | 0 \rangle \\ &= \delta(x^0 - y^0) \langle 0 | [\partial_t \phi(x), \phi^*(y)] | 0 \rangle + 2ieA_0 \delta(x^0 - y^0) \langle 0 | [\phi(x), \phi^*(y)] | 0 \rangle \end{aligned}$$

We used the distributional identity $\partial_x \theta(x) = \delta(x)$. Using the equal-time commutation relations, we notice that $2ieA_0 \delta(x^0 - y^0) \langle 0 | [\phi(x), \phi^*(y)] \rangle = 0$. It also follows that

$$\begin{aligned} \delta(x^0 - y^0) \langle 0 | [\partial_t \phi(x), \phi^*(y)] | 0 \rangle &= \delta(x^0 - y^0) \langle 0 | [\partial_t \phi(x) + ieA_0(x), \phi^*(y)] | 0 \rangle \\ &= \delta(x^0 - y^0) \langle 0 | [\pi^*(x), \phi^*(y)] | 0 \rangle \\ &= -i\delta(\vec{x} - \vec{y}) . \end{aligned}$$

We once again used the equal-time commutation relations. Therefore:

$$(D_\mu^2 + m^2) i\theta(x^0 - y^0) \langle 0 | [\phi(x), \phi^*(y)] | 0 \rangle = \delta(x - y) \quad (\text{A.2})$$

As Δ^- vanishes for $x^0 < y^0$, this shows that Δ^- indeed is the retarded fundamental solution of the Klein-Gordon operator. One analogously shows that the advanced propagator Δ^+ is given by:

$$\Delta^+ = -i\theta(x^0 - y^0) \langle 0 | [\phi(x), \phi^*(y)] | 0 \rangle \quad (\text{A.3})$$

These calculations especially show that the two-point functions fulfil the condition

$$\omega_2 \phi \phi^*(x, y) - \omega_2 \phi^* \phi(x, y) = i (\Delta^+ - \Delta^-) . \quad (\text{A.4})$$

Appendix B

The Charge Density as a Distribution

When using the summation of modes method, we calculated the sum

$$\sum_{n=1}^{\infty} \rho^I(z) = -\lambda \frac{e}{a} \sum_{n=1}^{\infty} z(1-z) \sin(2n\pi z) , \quad (\text{B.1})$$

in Abel's sense. Now, we however want to understand it in a distributional sense, which may be more appropriate. For this reason we introduce a test function $f(z)$ and consider the expression:

$$\sum_{n=1}^{\infty} \int_0^1 \rho^I(z) f(z) \, dz \quad (\text{B.2})$$

We integrate by parts by making use of the formula:

$$\int z(1-z) \sin(2n\pi z) \, dz = -\frac{(1-z)z \cos(2n\pi z)}{2\pi n} - \frac{\cos(2n\pi z)}{4\pi^3 n^3} + \frac{(1-2z) \sin(2n\pi z)}{4\pi^2 n^2} \quad (\text{B.3})$$

Thus:

$$\begin{aligned} & \sum_{n=1}^{\infty} \int_0^1 z(1-z) \sin(2n\pi z) f(z) \, dz \\ &= \sum_{n=1}^{\infty} \int_0^1 dz f'(z) \left\{ \frac{(1-z)z \cos(2n\pi z)}{2\pi n} + \frac{\cos(2n\pi z)}{4\pi^3 n^3} - \frac{(1-2z) \sin(2n\pi z)}{4\pi^2 n^2} \right\} \end{aligned}$$

Using

$$\int z(1-z) \cos(2n\pi z) \, dz = \frac{\sin(2n\pi z)}{4\pi^3 n^3} + \frac{z(1-z) \sin(2n\pi z)}{2\pi n} - \frac{(2z-1) \cos(2n\pi z)}{4\pi^2 n^2} , \quad (\text{B.4})$$

we can perform partial integration on the first term again, one gets:

$$\begin{aligned}
& \sum_{n=1}^{\infty} \int_0^1 z(1-z) \sin(2n\pi z) f(z) \, dz \\
&= \sum_{n=1}^{\infty} \int_0^1 dz \left(\left[-\frac{\sin(2n\pi z)}{8\pi^4 n^4} - \frac{z(1-z) \sin(2n\pi z)}{4\pi^2 n^2} + \frac{(2z-1) \cos(2n\pi z)}{8\pi^3 n^3} \right] f''(z) \right. \\
&\quad \left. + \left[\frac{\cos(2n\pi z)}{4\pi^3 n^3} - \frac{(1-2z) \sin(2n\pi z)}{4\pi^2 n^2} \right] f'(z) \right)
\end{aligned}$$

We integrate the terms containing $\frac{1}{n^3}$ and $\frac{1}{n^4}$ by parts "backwards":

$$\begin{aligned}
& \sum_{n=1}^{\infty} \int_0^1 z(1-z) \sin(2n\pi z) f(z) \, dz \\
&= \sum_{n=1}^{\infty} \int_0^1 dz \left(\frac{\sin(2n\pi z)}{2\pi^2 n^2} f(z) - \frac{z(1-z) \sin(2n\pi z)}{4\pi^2 n^2} f''(z) \right. \\
&\quad \left. + \frac{(2z-1) \sin(2n\pi z)}{4\pi^2 n^2} f'(z) - \frac{\sin(2n\pi z)}{2\pi^2 n^2} f(z) + \frac{\sin(2n\pi z)}{2\pi^2 n^2} f(z) - \frac{(1-2z) \sin(2n\pi z)}{4\pi^2 n^2} f'(z) \right) \\
&= \sum_{n=1}^{\infty} \int_0^1 dz \left(\frac{\sin(2n\pi z)}{2\pi^2 n^2} f(z) - \frac{z(1-z) \sin(2n\pi z)}{4\pi^2 n^2} f''(z) + \frac{(2z-1) \sin(2n\pi z)}{2\pi^2 n^2} f'(z) \right)
\end{aligned}$$

As each term is absolutely converging we can interchange the sum and the integral. We further use $\sin(2n\pi z) = \frac{1}{2i}(e^{2i\pi n z} - e^{-2i\pi n z})$ and (cf. [20])

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = - \int_0^x \frac{\ln(1-t)}{t} \, dt. \quad (\text{B.5})$$

The above function is known as the *dilogarithmic function* (cf. [20]). Thus:

$$\begin{aligned}
& \sum_{n=1}^{\infty} \int_0^1 z(1-z) \sin(2n\pi z) f(z) \, dz \\
&= \int_0^1 dz \frac{1}{4i\pi^2} \left(-f(z) - (2z-1)f'(z) + \frac{z(1-z)}{2} f''(z) \right) \left(\int_0^{e^{2i\pi z}} \frac{\ln(1-t)}{t} \, dt - \int_0^{e^{-2i\pi z}} \frac{\ln(1-t)}{t} \, dt \right)
\end{aligned}$$

Further

$$\begin{aligned}
& \frac{\partial}{\partial z} \left(\int_0^{e^{2i\pi z}} \frac{\ln(1-t)}{t} \, dt - \int_0^{e^{-2i\pi z}} \frac{\ln(1-t)}{t} \, dt \right) = 2\pi i (\ln(1 - e^{2i\pi z}) + \ln(1 - e^{-2i\pi z})) \\
&= 4\pi i \ln |2 \sin(\pi z)|
\end{aligned}$$

We now want to integrate by parts backwards. As $-1 - (-1)\frac{\partial}{\partial z}(2z-1) + \frac{\partial^2}{\partial z^2} \frac{z(1-z)}{2} = 0$ the term involving the dilogarithmic function cancels and only terms containing its derivatives remain. Hence:

$$\begin{aligned}
& \sum_{n=1}^{\infty} \int_0^1 z(1-z) \sin(2n\pi z) f(z) \, dz \\
&= \int_0^1 dz f(z) \frac{1}{4i\pi^2} \left((2z-1)4\pi i \ln |2 \sin(\pi z)| + \right. \\
&\quad \left. + 2 \frac{1-2z}{2} 4\pi i \ln |2 \sin(\pi z)| + \frac{z(1-z)}{2} \frac{\partial}{\partial z} 4\pi i \ln |2 \sin(\pi z)| \right) \\
&= \int_0^1 dz f(z) \left(-\frac{1}{2} z(1-z) \cot(\pi z) \right)
\end{aligned}$$

By comparing this to our original expression, we might thus think of the charge density as a distribution of the form:

$$\hat{\rho}(z) = -\lambda \frac{e}{a} \frac{1}{2} z(1-z) \cot(\pi z) \quad (\text{B.6})$$

We see that this result coincides with the Abel limit.

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