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Relativistic Quantum Field Theory — Major Homework Exercise

5 pages — Problems 1 to 3

Starred exercises are bonus questions.

Problem 1: The energy-momentum tensor

We consider a real scalar field, subject to the action

$$S = \int \mathcal{L} \mathrm{d}^4 x = \int \left(-\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 \right) \mathrm{d}^4 x \tag{0.1}$$

a) Derive the Euler-Lagrange equations of motion for ϕ.

b) The action (0.1) is invariant under translations. Infinitesimally, these are given by

$$\phi(x) \mapsto \phi(x) + \varepsilon^\mu \partial_\mu \phi(x),$$

with ε^μ an infinitesimal displacement vector. Show that the Noether current associated to this symmetry is

$$T^\mu[\varepsilon] = T^\mu_\nu \varepsilon^\nu$$

with the *energy-momentum tensor*

$$T^\mu_\nu = -\partial^\mu \phi \partial_\nu \phi + \frac{1}{2}\delta^\mu_\nu \left(\partial_\lambda \phi \partial^\lambda \phi + m^2 \phi^2 \right).$$

c) Show that the energy-momentum tensor is conserved on-shell, i.e.

$$\partial_\mu T^\mu_\nu = 0$$

for all ϕ satisfying the equation of motion.

d) Given the representation

$$\phi(x) = \int \widetilde{\mathrm{d}k} \left(a(\underline{k}) e^{ikx} + a(\underline{k})^* e^{-ikx} \right)$$

with

$$\widetilde{\mathrm{d}k} = \frac{\mathrm{d}^3 k}{2\omega(2\pi)^3}, \qquad \omega = \sqrt{\underline{k}^2 + m^2}, \qquad kx = -\omega t + \underline{k} \cdot \underline{x},$$

write the Hamiltonian and the momentum,

$$H = P_0 = \int_{x^0=0} {:}T^0_0{:}\, \mathrm{d}^3 x, \qquad P_i = \int_{x^0=0} {:}T^0_i{:}\, \mathrm{d}^3 x$$

in terms of a, a^*. Here $: \cdot :$ denotes normal ordering.

e) Using canonical commutation relations

$$[a(\underline{k}), a^*(\underline{q})] = (2\pi)^3 2\omega \delta(\underline{k} - \underline{q}),$$

verify that these operators indeed generate time and spatial translations, i.e.,

$$i[H, \phi(x)] = \dot{\phi}(x), \qquad i[P_i, \phi(x)] = \partial_i \phi(x).$$

f*) Instead of translations, we now consider rotations. Infinitesimally, a rotation around $\underline{e}_i$ is given by

$$\phi(x) \mapsto \phi(x) - \omega \varepsilon_{ijk} x_j \partial_k \phi(x),$$

with ω infinitesimal. Show that the Noether current associated to this symmetry is

$$\ell_i^\mu = \varepsilon_{ijk} \left(\partial^\mu \phi x_j \partial_k \phi - \frac{1}{2} \delta_k^\mu x_j \left(\partial_\lambda \phi \partial^\lambda \phi + m^2 \phi^2 \right) \right).$$

g*) Express the corresponding charges

$$L_i = \int_{x^0=0} \ell_i^0 \mathrm{d}^3 x$$

in terms of a, a^*.

h*) Verify that the generators form a representation of the rotation algebra, i.e.,

$$[L_i, L_j] = i\varepsilon_{ijk} L_k.$$

Problem 2: $e^-\varphi \to e^-\varphi$ scattering in Yukawa theory

We consider the scattering of an electron (mass m) off a scalar (mass M) in Yukawa theory, i.e., subject to the coupling $g\varphi\bar{\Psi}\Psi$. We denote the in (out) electron momentum by p (p') and the in (out) scalar momentum by k (k') and the in (out) electron spin by s (s'). We want to express the final result in terms of the Mandelstam variables [Do not confuse the Mandelstam s with the spin s!]

$$s = -(p+k)^2, \qquad t = -(p-p')^2, \qquad u = -(p-k')^2.$$

a) Verify the kinematical relations

$$p \cdot k = p' \cdot k' = -\tfrac{1}{2}(s - m^2 - M^2), \qquad p \cdot p' = \tfrac{1}{2}(t - 2m^2),$$
$$p \cdot k' = p' \cdot k = +\tfrac{1}{2}(u - m^2 - M^2), \qquad k \cdot k' = \tfrac{1}{2}(t - 2M^2)$$

and

$$u + s + t = 2m^2 + 2M^2.$$

b) Draw the tree-level Feynman diagrams contributing to $e^-\varphi \to e^-\varphi$ scattering in Yukawa theory.

c) Evaluate the Feynman diagrams and determine an expression for the scattering amplitude $\mathcal{T}$.

d) Give an expression for the modulus square $|\mathcal{T}|^2$ of the scattering amplitude.

e) If unpolarized electrons are used, we should sum over final spins s' and average over initial spins s. Show that this average can be written as

$$\langle|\mathcal{T}|^2\rangle = \frac{1}{2}\sum_{s,s'}|\mathcal{T}|^2 = \frac{1}{2}\,\mathrm{Tr}\left[(-\not{p}'+m)A(-\not{p}+m)A\right]$$

with

$$A = g^2\left(\frac{-\not{k}+2m}{m^2-s}+\frac{\not{k}'+2m}{m^2-u}\right).$$

Hint: Use the spin sum

$$\sum_{s=\pm} u_s(\underline{p})\bar{u}_s(\underline{p}) = -\not{p}+m$$

and the relation

$$(\not{p}+m)u_s(\underline{p}) = 0.$$

f) Rewrite the above as

$$\langle|\mathcal{T}|^2\rangle = g^4\left[\frac{\langle\Phi_{ss}\rangle}{(m^2-s)^2}+\frac{\langle\Phi_{su}\rangle+\langle\Phi_{us}\rangle}{(m^2-s)(m^2-u)}+\frac{\langle\Phi_{uu}\rangle}{(m^2-u)^2}\right]$$

with

$$\begin{aligned}
\langle\Phi_{ss}\rangle &= \tfrac{1}{2}\,\mathrm{Tr}\left[(-\not{p}'+m)(-\not{k}+2m)(-\not{p}+m)(-\not{k}+2m)\right],\\
\langle\Phi_{su}\rangle &= \tfrac{1}{2}\,\mathrm{Tr}\left[(-\not{p}'+m)(-\not{k}+2m)(-\not{p}+m)(+\not{k}'+2m)\right],\\
\langle\Phi_{us}\rangle &= \tfrac{1}{2}\,\mathrm{Tr}\left[(-\not{p}'+m)(+\not{k}'+2m)(-\not{p}+m)(-\not{k}+2m)\right],\\
\langle\Phi_{uu}\rangle &= \tfrac{1}{2}\,\mathrm{Tr}\left[(-\not{p}'+m)(+\not{k}'+2m)(-\not{p}+m)(+\not{k}'+2m)\right].
\end{aligned}$$

and show that one obtains

$$\begin{aligned}
\langle\Phi_{ss}\rangle &= -su+m^2(9s+u)+7m^4-8m^2M^2+M^4,\\
\langle\Phi_{su}\rangle &= +su+3m^2(s+u)+9m^4-8m^2M^2-M^4,\\
\langle\Phi_{us}\rangle &= +su+3m^2(s+u)+9m^4-8m^2M^2-M^4,\\
\langle\Phi_{uu}\rangle &= -su+m^2(s+9u)+7m^4-8m^2M^2+M^4.
\end{aligned}$$

Hint: Use

$$\begin{aligned}
\mathrm{Tr}[1] &= 4,\\
\mathrm{Tr}[\not{k}] &= 0\\
\mathrm{Tr}[\not{k}\not{l}] &= -4k\cdot l,\\
\mathrm{Tr}[\not{k}\not{l}\not{p}] &= 0,\\
\mathrm{Tr}[\not{k}\not{l}\not{p}\not{q}] &= 4[(k\cdot q)(l\cdot p)-(k\cdot p)(l\cdot q)+(k\cdot l)(p\cdot q)]
\end{aligned}$$

and the fact that under $k \leftrightarrow -k'$ we have $\langle\Phi_{ss}\rangle \leftrightarrow \langle\Phi_{uu}\rangle$ and $\langle\Phi_{su}\rangle \leftrightarrow \langle\Phi_{us}\rangle$.

We want to compare with Compton scattering. We take the limit in which the scalar particle is massless, $M = 0$, and consider the electron initially at rest.

g) Verify the relation

$$\cos\theta = 1 - m\left(\frac{1}{E'} - \frac{1}{E}\right).$$

h) Show that the relativistically invariant two-body phase space reduces to

$$(2\pi)^4 \widetilde{\mathrm{d}p'}\widetilde{\mathrm{d}k'}\delta^{(4)}(p + k - p' - k') = \frac{\mathrm{d}\cos\theta}{8\pi}\frac{E'^2}{mE}$$

with θ the scattering angle of the scalar particle and $E' = E_{k'}$, $E = E_k$.

i) Compute the differential cross section

$$\mathrm{d}\sigma = \frac{1}{4mE}(2\pi)^4 \widetilde{\mathrm{d}p'}\widetilde{\mathrm{d}k'}\delta^{(4)}(p + k - p' - k')\langle|\mathcal{T}|^2\rangle$$

and write it in the form

$$\mathrm{d}\sigma = \frac{g^4}{32\pi m^2}\frac{E'^2}{E^2}f(E, E', \theta)\ \mathrm{d}\cos\theta.$$

Hint: Use the relations derived in a) and g). Show that in the limit of small energies, $E \ll m$,

$$\mathrm{d}\sigma = \frac{g^4}{8\pi m^2}\cos^2\theta\ \mathrm{d}\cos\theta.$$

Compare with Thomson's result

$$\mathrm{d}\sigma = \frac{e^4}{16\pi m^2}(1 + \cos^2\theta)\mathrm{d}\cos\theta$$

for the classical differential cross section in Compton scattering. Could the photon be a scalar?

Problem 3: Loop corrections in ϕ^4 theory

We consider a real scalar field, with ϕ^4 interaction:

$$S = \int \mathcal{L}\mathrm{d}^4x = \int \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\right)\mathrm{d}^4x.$$

a) Draw all diagrams which contribute, at the one-loop level, to the vertex correction $-iV_4(p_1, p_2, p_3, p_4)$.

b) Write the one-loop vertex correction in the form

$$V_4(p_1, p_2, p_3, p_4)/\lambda = \Sigma(p_1 + p_2) + \Sigma(p_1 + p_3) + \Sigma(p_1 + p_4)$$

with

$$\Sigma(k) = \frac{i\lambda}{2}\int \frac{\mathrm{d}^d\ell}{(2\pi)^d}F(k, \ell).$$

c) Use Feynman's identity

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{(xA + (1 - x)B)^2}$$

to write $F(k, \ell)$ as

$$F(k, \ell) = \int_0^1 \mathrm{d}x \frac{1}{(q^2 + D_\varepsilon)^2}$$

with

$$q = \ell + xk, \qquad D_\varepsilon = x(1 - x)k^2 + m^2 - i\varepsilon.$$

d) Use Wick rotation to write

$$\Sigma(k^2) = -\frac{\lambda}{2}\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d\bar{q}}{(2\pi)^d}\frac{1}{(\bar{q}^2 + D_0)^2},$$

with $\bar{q}^2$ the Euclidean square.

e) Show that in $d = 4 - \varepsilon$ dimensions, λ must have mass dimension ε. Hence, to keep λ dimensionless, replace $\lambda \to \lambda\mu^\varepsilon$, with some constant μ of mass dimension 1.

f) Use

$$\Gamma(n+1) = n!, \qquad \Gamma(-n + x) = \frac{(-1)^n}{n!}\left(\frac{1}{x} - \gamma + \sum_{k=1}^{n} k^{-1} + \mathcal{O}(x)\right),$$

the area

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{1}{2}d)}$$

of the unit sphere in d dimensions,

$$\int_0^\infty \mathrm{d}q \frac{q^{2a-1}}{(q^2 + D)^b} = \frac{\Gamma(a)\Gamma(b-a)}{2\Gamma(b)} D^{-b+a},$$

and

$$A^{\frac{\varepsilon}{2}} = 1 + \frac{\varepsilon}{2}\log A + \mathcal{O}(\varepsilon^2)$$

to write Σ in the form

$$\Sigma(k^2) = \frac{1}{\varepsilon}A + B + \mathcal{O}(\varepsilon).$$

Does A depend on k^2? How can one thus handle the $\frac{1}{\varepsilon}$ singularity?