Prof. Dr. S. Hollands
Prof. Dr. R. Verch
Dr. J. Zahn
Dr. M. Schmidt

UNIVERSITÄT LEIPZIG

Inst. f. Theoretische Physik
Winter Term 2017/18

Relativistic Quantum Field Theory — Problem Sheet 9

2 pages — Problems 9.1 to 9.2

Problem 9.1: γ matrix algebra

a) Show that, for γ matrices fulfilling

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\mathbb{1},$$

one obtains with

$$S^{\mu\nu} := \tfrac{i}{4}[\gamma^\mu, \gamma^\nu]$$

representation matrices of the Lorentz algebra, i.e.,

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(\eta^{\mu\rho}S^{\nu\sigma} - \eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\sigma}S^{\nu\rho} + \eta^{\nu\sigma}S^{\mu\rho}\right).$$

Show that, in the representation

$$\gamma^0 := \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^i := \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{0.1}$$

with σ_i the Pauli matrices, one has

$$S^{ij} = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \qquad S^{i0} = \frac{i}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}.$$

b) One defines

$$\beta := \gamma^0 = \beta^{-1}, \qquad \gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = -\tfrac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

where $\varepsilon_{0123} = -1$. Check that

$$\gamma_5^2 = \mathbb{1}, \qquad \{\gamma_5, \gamma^\mu\} = 0, \qquad [S^{\mu\nu}, \gamma_5] = 0, \qquad \beta\gamma_5\beta^{-1} = -\gamma_5.$$

The last two properties imply that γ_5 is a pseudo-scalar. Show that in the representation (0.1)

$$\gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}.$$

c) Check that, in the representation (0.1),

$$\mathcal{C} := -i\gamma^2\gamma^0 = \begin{pmatrix} -\varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix} = -\mathcal{C}^{-1},$$

with

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Check that, in the representation (0.1)

$$\beta\gamma^\mu\beta^{-1} = (\gamma^\mu)^\dagger, \qquad \mathcal{C}\gamma^\mu\mathcal{C}^{-1} = -(\gamma^\mu)^T, \qquad \beta\mathcal{C}\gamma^\mu\mathcal{C}^{-1}\beta^{-1} = -(\gamma^\mu)^*.$$

Here $\dagger$ denotes the adjoint and $*$ the complex conjugate. [Hint: The third equality follows easily from the first two upon noting that $\beta^T = \beta$.] Use this to prove

$$\begin{aligned} \beta S^{\mu\nu}\beta^{-1} &= (S^{\mu\nu})^\dagger, & \beta\gamma_5\beta^{-1} &= -(\gamma_5)^\dagger, \\ \mathcal{C}S^{\mu\nu}\mathcal{C}^{-1} &= -(S^{\mu\nu})^T, & \mathcal{C}\gamma_5\mathcal{C}^{-1} &= (\gamma_5)^T, \\ \beta\mathcal{C}S^{\mu\nu}\mathcal{C}^{-1}\beta^{-1} &= -(S^{\mu\nu})^*, & \beta\mathcal{C}\gamma_5\mathcal{C}^{-1}\beta^{-1} &= -(\gamma_5)^*. \end{aligned}$$

Problem 9.2: Polarization spinors

In the lecture, we defined

$$u_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_+(\mathbf{0}) = \sqrt{m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

and

$$u_s(\mathbf{p}) = e^{i\eta\hat{\mathbf{p}}\cdot\mathbf{K}}u_s(\mathbf{0}), \qquad v_s(\mathbf{p}) = e^{i\eta\hat{\mathbf{p}}\cdot\mathbf{K}}v_s(\mathbf{0}),$$

with $\eta = \operatorname{arcsinh}\frac{|\mathbf{p}|}{m}$ the rapidity, $\hat{\mathbf{p}} = \frac{\mathbf{p}}{|\mathbf{p}|}$, and $K^j = S^{0j}$. We also had

$$\bar{u}_s(\mathbf{p}) := u_s^\dagger(\mathbf{p})\beta = \bar{u}_s(\mathbf{0})e^{-i\eta\hat{\mathbf{p}}\cdot\mathbf{K}}, \qquad \bar{v}_s(\mathbf{p}) := v_s^\dagger(\mathbf{p})\beta = \bar{v}_s(\mathbf{0})e^{-i\eta\hat{\mathbf{p}}\cdot\mathbf{K}}.$$

a) Show that

$$e^{i\eta\hat{\mathbf{p}}\cdot\mathbf{K}} = \begin{pmatrix} e^{\frac{1}{2}\eta\hat{\mathbf{p}}\cdot\sigma} & 0 \\ 0 & e^{-\frac{1}{2}\eta\hat{\mathbf{p}}\cdot\sigma} \end{pmatrix}$$

and consequently

$$e^{-i\eta\hat{\mathbf{p}}\cdot\mathbf{K}}\gamma^0 e^{-i\eta\hat{\mathbf{p}}\cdot\mathbf{K}} = \gamma^0.$$

From this, conclude that

$$\begin{aligned} \bar{u}_{s'}(\mathbf{p})\gamma^0 v_s(-\mathbf{p}) &= 0, \\ \bar{v}_{s'}(\mathbf{p})\gamma^0 u_s(-\mathbf{p}) &= 0. \end{aligned}$$

b) Check that

$$\sum_{s=\pm} u_s(\mathbf{0})\bar{u}_s(\mathbf{0}) = m\gamma^0 + m, \qquad \sum_{s=\pm} v_s(\mathbf{0})\bar{v}_s(\mathbf{0}) = m\gamma^0 - m,$$

and conclude that

$$\sum_{s=\pm} u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = -\not{p} + m, \qquad \sum_{s=\pm} v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) = -\not{p} - m.$$