Problem 8.1: Loop corrections to the propagator

In the lecture, we derived the self-energy

\[ \Pi(p^2) = \alpha \frac{1}{2} \int_0^1 dx \ D \ln \frac{D}{D_0} + C(p^2 + m^2) \]

where \( D = x(1-x)p^2 + m^2 \) and \( D_0 = D|_{p^2 = -m^2} \). Fix the constant \( C \) by requiring that \( \Pi'(-m^2) = 0 \). From the requirement that this equals

\[ \alpha \left( \frac{1}{2} \int_0^1 dx \ D \ln \frac{D}{m^2} + \frac{1}{3} \kappa_A p^2 + 2 \kappa_B m^2 \right) \]

determine the renormalization constants \( \kappa_A \) and \( \kappa_B \).

Problem 8.2: Loop corrections to the vertex

We want to compute the one-loop contribution to the three-point vertex with incoming momenta \( k_1, k_2, k_3 \). To be precise, we define \(-iV_3(k_1, k_2, k_3)\) as the sum over all 1PI diagrams with external momenta \( k_1, k_2, k_3 \) (with momentum conservation taken into account).

a) Show that

\[ \tilde{\Delta}(l - k_1)\tilde{\Delta}(l + k_2)\tilde{\Delta}(l) = \int dF_3 (q^2 + D)^{-3} \]

with

\[ \int dF_3 = 2 \int_0^1 dx_1 dx_2 dx_3 \ \delta(1-x_1-x_2-x_3), \]

\[ q = l - x_1 k_1 - x_2 k_2, \]

\[ D = x_3 x_1 k_1^2 + x_3 x_2 k_2^2 + x_1 x_2 k_3^2 + m^2 - i\varepsilon. \]

Hint: Use Feynman’s formula and momentum conservation.

b) Show that

\[ (-i)^3 (-i\lambda)^3 \int \frac{d^d l}{(2\pi)^d} \tilde{\Delta}(l - k_1)\tilde{\Delta}(l + k_2)\tilde{\Delta}(l) = -i\lambda^3 \int dF_3 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^3}, \]

where \( q^2 = q_1^2 + \cdots + q_d^2 \) is the Euclidean square. Hint: Perform a change of variables and a Wick rotation.
c) Using
\[ \int_0^\infty dq \frac{q^{2a-1}}{(q^2 + D)^b} = \frac{\Gamma(a)\Gamma(b-a)}{2\Gamma(b)} D^{-b+a}, \]
show that
\[ \int \frac{d^d \bar{q}}{(2\pi)^d (\bar{q}^2 + D)^3} = \frac{\Gamma(3 - \frac{d}{2})}{2(4\pi)^{\frac{d}{2}}} D^{-3 + \frac{d}{2}}. \]

d) Argue that the one-loop vertex correction is finite for \( d < 6 \).

e) We set \( d = 6 - \varepsilon \) and replace \( \lambda \) by \( \lambda \tilde{\mu}^\varepsilon \) with \( \tilde{\mu} \) some mass, in order to keep \( \lambda \) dimensionless. Show that the vertex function is given by
\[ V_3(k_1, k_2, k_3) = \lambda \left( 1 + \frac{\alpha}{2} \Gamma\left(\frac{\varepsilon}{2}\right) \int dF_3 \left( \frac{4\pi \tilde{\mu}^2}{D} \right)^\varepsilon + O(\alpha^2) \right), \]
with \( \alpha = \frac{\lambda^2}{(4\pi)^\gamma} \).

f) Show that in the limit \( \varepsilon \to 0 \), we get
\[ V_3(k_1, k_2, k_3) = \lambda \left( 1 + \alpha \left[ \frac{1}{\varepsilon} + \ln \frac{\mu}{m} - \frac{1}{2} \int dF_3 \ln \frac{D}{m^2} \right] + O(\alpha^2) \right), \]
where \( \mu^2 = 4\pi e^{-\gamma} \tilde{\mu}^2 \) with \( \gamma \) the Euler-Mascheroni constant. Hint: Use \( \int dF_3 = 1 \).