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Relativistic Quantum Field Theory — Problem Sheet 8

2 pages — Problems 8.1 to 8.2

Problem 8.1: Loop corrections to the propagator

In the lecture, we derived the self-energy

$$\Pi(p^2) = \frac{\alpha}{2} \int_0^1 \mathrm{d}x \; D \ln \frac{D}{D_0} + C(p^2 + m^2)$$

where $D = x(1-x)p^2 + m^2$ and $D_0 = D|_{p^2=-m^2}$. Fix the constant C by requiring that $\Pi'(-m^2) = 0$. From the requirement that this equals

$$\alpha \left(\frac{1}{2} \int_0^1 \mathrm{d}x \; D \ln \frac{D}{m^2} + \frac{1}{3} \kappa_A p^2 + 2\kappa_B m^2 \right)$$

determine the renormalization constants κ_A and κ_B.

Problem 8.2: Loop corrections to the vertex

We want to compute the one-loop contribution to the three-point vertex with incoming momenta k_1, k_2, k_3. To be precise, we define $-iV_3(k_1, k_2, k_3)$ as the sum over all 1PI diagrams with external momenta k_1, k_2, k_3 (with momentum conservation taken into account).

a) Show that

$$\tilde{\Delta}(l - k_1)\tilde{\Delta}(l + k_2)\tilde{\Delta}(l) = \int \mathrm{d}F_3 \left(q^2 + D\right)^{-3}$$

with

$$\begin{aligned} \int \mathrm{d}F_3 &= 2 \int_0^1 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \; \delta(1 - x_1 - x_2 - x_3), \\ q &= l - x_1 k_1 - x_2 k_2, \\ D &= x_3 x_1 k_1^2 + x_3 x_2 k_2^2 + x_1 x_2 k_3^2 + m^2 - i\varepsilon. \end{aligned}$$

Hint: Use Feynman's formula and momentum conservation.

b) Show that

$$(-i)^3(-i\lambda)^3 \int \frac{\mathrm{d}^d l}{(2\pi)^d} \tilde{\Delta}(l - k_1)\tilde{\Delta}(l + k_2)\tilde{\Delta}(l) = -i\lambda^3 \int \mathrm{d}F_3 \int \frac{\mathrm{d}^d \bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2 + D)^3},$$

where $\bar{q}^2 = q_1^2 + \cdots + q_d^2$ is the Euclidean square. Hint: Perform a change of variables and a Wick rotation.

c) Using

$$\int_0^\infty \mathrm{d}q \; \frac{q^{2a-1}}{(q^2+D)^b} = \frac{\Gamma(a)\Gamma(b-a)}{2\Gamma(b)} D^{-b+a},$$

show that

$$\int \frac{\mathrm{d}^d\bar{q}}{(2\pi)^d} \frac{1}{(\bar{q}^2+D)^3} = \frac{\Gamma(3-\frac{d}{2})}{2(4\pi)^{\frac{d}{2}}} D^{-3+\frac{d}{2}}.$$

d) Argue that the one-loop vertex correction is finite for $d < 6$.

e) We set $d = 6 - \varepsilon$ and replace λ by $\lambda\tilde{\mu}^{\frac{\varepsilon}{2}}$ with $\tilde{\mu}$ some mass, in order to keep λ dimensionless. Show that the vertex function is given by

$$V_3(k_1, k_2, k_3) = \lambda \left(1 + \frac{\alpha}{2}\Gamma(\frac{\varepsilon}{2}) \int \mathrm{d}F_3 \left(\frac{4\pi\tilde{\mu}^2}{D}\right)^{\frac{\varepsilon}{2}} + \mathcal{O}(\alpha^2) \right),$$

with $\alpha = \frac{\lambda^2}{(4\pi^3)}$.

f) Show that in the limit $\varepsilon \to 0$, we get

$$V_3(k_1, k_2, k_3) = \lambda \left(1 + \alpha \left[\frac{1}{\varepsilon} + \ln\frac{\mu}{m} - \frac{1}{2} \int \mathrm{d}F_3 \ln\frac{D}{m^2} \right] + \mathcal{O}(\alpha^2) \right),$$

where $\mu^2 = 4\pi e^{-\gamma}\tilde{\mu}^2$ with γ the Euler-Mascheroni constant. Hint: Use $\int \mathrm{d}F_3 = 1$.