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Winter Term 2017/18

Relativistic Quantum Field Theory — Problem Sheet 6

3 pages — Problems 6.1 to 6.4

Problem 6.1: Scattering off a solid sphere

Consider the classical, non-relativistic scattering of a point-like particle off a solid sphere of radius R.

a) Find the impact parameter b as a function of the scattering angle θ.

b) Determine the differential cross section

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b\mathrm{d}b}{\sin\theta\mathrm{d}\theta}.$$

c) Determine the total cross section

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega.$$

To which area does it correspond?

Problem 6.2: Compton scattering

In $2 \to 2$ scattering, $p_A, p_B \to p_1, p_2$, one typically uses the Lorentz invariant Mandelstam variables

$$s = -(p_A + p_B)^2, \qquad t = -(p_A - p_1)^2, \qquad u = -(p_A - p_2)^2.$$

a) Verify

$$s = -(p_1 + p_2)^2, \qquad t = -(p_B - p_2)^2, \qquad u = -(p_B - p_1)^2.$$

and

$$s + t + u = m_A^2 + m_B^2 + m_1^2 + m_2^2$$

b) We will be considering Compton scattering, i.e., scattering of a photon (initial momentum p_A, final momentum p_1) off an electron (initial momentum p_B, final momentum p_2, mass m). We work in the FT frame, i.e., $\underline{p_B} = 0$. Show that, using rotation symmetry around the incidence axis, the relativistically invariant two-body phase space reduces to

$$(2\pi)^4 \widetilde{\mathrm{d}p_1}\widetilde{\mathrm{d}p_2}\delta^{(4)}(p_A + p_B - p_1 - p_2) = \frac{\mathrm{d}\cos\theta}{8\pi}\frac{E_1^2}{mE_A},$$

where θ is the scattering angle of the photon and

$$\widetilde{\mathrm{d}p} = \frac{\mathrm{d}^3 p}{(2\pi)^3 2E}, \qquad p^\mu = (E_{\underline{p}}, \underline{p}).$$

c) Verify the relations

$$s = m^2 + 2mE_A, \qquad u = m^2 - 2mE_1, \qquad \cos\theta = 1 - m\left(\frac{1}{E_1} - \frac{1}{E_A}\right).$$

d) For the corresponding matrix element (after summing over spins and polarizations), one computes

$$|\mathcal{T}|^2 = 32\pi^2\alpha^2 \left(\frac{m^4 + m^2(3s+u) - su}{(m^2-s)^2} + \frac{m^4 + m^2(3u+s) - su}{(m^2-u)^2} + \frac{2m^2(s+u+2m^2)}{(m^2-s)(m^2-u)}\right),$$

with α the fine structure constant. Show that the differential cross section in the FT frame can be expressed by the Klein-Nishina formula

$$\mathrm{d}\sigma = \frac{\pi\alpha^2}{m^2}\frac{E_1^2}{E_A^2}\left(\frac{E_A}{E_1} + \frac{E_1}{E_A} - \sin^2\theta\right)\mathrm{d}\cos\theta$$

e) Verify that in the limit of small energies, $E_A \ll m$, one has $E_1 = E_A(1+\mathcal{O}(E_A/m))$. Conclude that in the limit $E_A \to 0$, one recovers the classical result of Thomson,

$$\mathrm{d}\sigma = \frac{\pi\alpha^2}{m^2}\left(1+\cos^2\theta\right)\mathrm{d}\cos\theta.$$

Problem 6.3: Feynman graphs and scattering amplitudes

Consider a theory of three real scalar fields A, B, C, with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu A\partial^\mu A + m_A^2 A^2 + \partial_\mu B\partial^\mu B + m_B^2 B^2 + \partial_\mu C\partial^\mu C + m_C^2 C^2\right) - \lambda ABC.$$

In the corresponding Feynman graphs, one may distinguish the propagators for the three fields by plain, dashed and wiggled lines. The vertex always has three lines of different types entering. Draw the tree level diagrams contributing to the processes

$$AA \to AA, \quad AA \to AB, \quad AA \to BB, \quad AA \to BC, \quad AB \to AB, \quad AB \to AC$$

and write the corresponding scattering amplitudes in the form

$$\mathcal{T} = \lambda^2\left(\frac{c_s}{m_s^2 - s} + \frac{c_t}{m_t^2 - t} + \frac{c_u}{m_u^2 - u}\right).$$

In case a process is not possible at tree level, try to find a one-loop diagram contributing to the scattering amplitude.

Problem 6.4: The square of the two-point function

We want to show that the square $w_2(x)^2$ of the two-point function

$$w_2(x) = \int \widetilde{\mathrm{d}k}\, e^{ikx}$$

of a scalar field of mass m is well defined.

a) Write the Fourier transform of w_2^2 in the form

$$\widetilde{w_2^2}(k) = \int \mathrm{d}^4 x e^{-ikx} w_2^2(x) = \int \widetilde{\mathrm{d}p} \widetilde{\mathrm{d}l} f(k-p-l)$$

for some distribution f.

b) Argue that $\widetilde{w_2^2}(k)$ has support above the upper mass shell $2m$, i.e., for $-k^2 \geq 4m^2$, $k^0 > 0$.

c) Argue that, by Lorentz invariance, it thus suffices to consider $k = (M, 0)$. Perform the integrations over $\underline{p}$ and $\underline{l}$ and show that

$$\widetilde{w_2^2}(k) = \begin{cases} C \frac{\sqrt{-k^2-4m^2}}{\sqrt{-k^2}} & -k^2 > 4m^2, k^0 > 0, \\ 0 & \text{otherwise,} \end{cases}$$

for some constant C.

This is a tempered distribution, so its inverse Fourier transform w_2^2 is a tempered distribution. The origin of the well-definedness is that $\tilde{w}_2(k)$ has support in the positive energy cone: The set of positive energy momenta p, l that add up to a fixed positive energy momentum k is compact, so the integration in momentum space is only over a compact subset.