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Relativistic Quantum Field Theory — Problem Sheet 5

2 pages — Problems 5.1 to 5.3

Problem 5.1

Suppose that a classical scalar field $\varphi(x)$ on Minkowski spacetime has a canonically conjugate momentum field $\pi(x)$ and a Hamiltonian density $\mathscr{H}(\varphi(x), \pi(x))$. Writing $\varphi_t(\underline{x}) = \varphi(t, \underline{x})$, $\pi_t(\underline{x}) = \pi(t, \underline{x})$, define $\mathtt{h}(\varphi_t, \pi_t) = \int d^3x\, \mathscr{H}(\varphi_t(\underline{x}), \pi_t(\underline{x}))$ and define $\frac{\delta\mathscr{H}}{\delta\varphi}(\varphi_t(\underline{x}), \pi_t(\underline{x}))$ by the condition that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathtt{h}(\varphi_t + \varepsilon\xi_t, \pi_t) = \int d^3x\, \xi_t(\underline{x}) \frac{\delta\mathscr{H}}{\delta\varphi}(\varphi_t(\underline{x}), \pi_t(\underline{x}))$$

for all variations $\xi_t(\underline{x})$ in $\mathcal{S}(\mathbb{R}^4)$ of the field configurations. Define $\frac{\delta\mathscr{H}}{\delta\pi}(\varphi_t(\underline{x}), \pi_t(\underline{x}))$ analogously. Let the Hamiltonian density be given by

$$\mathscr{H}(\varphi_t(\underline{x}), \pi_t(\underline{x})) = \frac{1}{2}\left(\pi_t(\underline{x})^2 + |\underline{\nabla}\varphi_t(\underline{x})|^2 + m^2\varphi_t(\underline{x})^2\right) + \frac{\lambda}{p}\varphi_t(\underline{x})^p$$

Show that the "Hamiltonian equations"

$$\dot{\varphi}_t(\underline{x}) = \frac{\delta\mathscr{H}}{\delta\pi}(\varphi_t(\underline{x}), \pi_t(\underline{x}))\,, \qquad \dot{\pi}_t(\underline{x}) = -\frac{\delta\mathscr{H}}{\delta\varphi}(\varphi_t(\underline{x}), \pi_t(\underline{x}))$$

imply the field equation

$$(\Box + m^2)\varphi(t, \underline{x}) + \lambda\varphi(t, \underline{x})^{p-1} = 0\,.$$

Problem 5.2

Let $\phi(t, \underline{x})$ be the field operators of the interaction-free quantized Klein-Gordon field and let $\pi(t, \underline{x}) = \dot{\phi}(t, \underline{x})$ the quantized canonically conjugate momentum field. These quantized field operators fulfill the equal-time commutation relations

$$[\phi(t, \underline{x}), \phi(t, \underline{y})] = 0 = [\pi(t, \underline{x}), \pi(t, \underline{y})]\,, \qquad [\phi(t, \underline{x}), \pi(t, \underline{y})] = i\delta(\underline{x} - \underline{y}) \cdot \mathbf{1}\,.$$

Let H_0 be the Hamilton operator of the free quantized Klein-Gordon field, so that $\dot{\phi}(t, \underline{x}) = i[H_0, \phi(t, \underline{x})]$ and $\dot{\pi}(t, \underline{x}) = i[H_0, \pi(t, \underline{x})]$.

Suppose that a quantized Klein-Gordon field with self-interaction has field operators $\Phi(t, \underline{x})$ and canonically conjugate momentum field operators $\Pi(t, \underline{x})$ and a Hamilton operator H so that $\dot{\Phi}(t, \underline{x}) = i[\mathsf{H}, \Phi(t, \underline{x})]$ and $\dot{\Pi}(t, \underline{x}) = i[\mathsf{H}, \Pi(t, \underline{x})]$. Furthermore, suppose that for any time t there is a unitary operator $U(t)$ (suitably differentiable in t) so that

$$\Phi(t, \underline{x}) = U(t)^{-1}\phi(t, \underline{x})U(t) \qquad \text{and} \qquad \Pi(t, \underline{x}) = U(t)^{-1}\pi(t, \underline{x})U(t)\,.$$

(a) Show that the operators $\Phi(t,\underline{x})$ and $\Pi(t,\underline{x})$ fulfill the same equal-time commutation relations as $\phi(t,\underline{x})$ and $\pi(t,\underline{x})$.

(b) Show that there is a number-valued function $E(t)$ so that

$$i\dot{U}(t)U(t)^{-1} = U(t)\mathsf{H}U(t)^{-1} - H_0 + E(t)\cdot\mathbf{1}\,.$$

Hint: You can use that any operator A that commutes for given t with all $\phi(t,\underline{x})$ and all $\pi(t,\underline{x})$ is a multiple of the unit operator.

(c) Can $E(t)$ be chosen to be equal to 0 by a suitable re-definition of $U(t)$?

Problem 5.3

In a Hilbert space $\mathcal{H}$, let $H(t) = H_0 + H_1(t)$ be a time-dependent family of Hamilton operators.

(a) Show that a family of Hilbert space vectors ψ_t, $t \in \mathbb{R}$ is a solution to

$$i\frac{d}{dt}\psi_t = H(t)\psi_t\,, \quad \psi_s = \chi$$

for any given Hilbert space vector χ if

$$\psi_t = \left(1 + \sum_{n=1}^{\infty}(-i)^n \int_s^t dt_1 \int_s^{t_1} \dots \int_s^{t_{n-1}} dt_n\, H(t_1)\cdots H(t_n)\right)\chi$$

(show this formally, assume the infinite sum converges).

(b) Show that the previous series formula (*Dyson series*) can be re-written in the form

$$\psi_t = \left(\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_s^t dt_1 \int_s^t \dots \int_s^t dt_n\, T\{H(t_1)\cdots H(t_n)\}\right)\chi$$

where $T\{H(t_1)\cdots H(t_n)\}$ is the time-ordered product of $H(t_1)\cdots H(t_n)$.

(c) Let

$$U(t,s) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_s^t dt_1 \int_s^t \dots \int_s^t dt_n\, T\{H(t_1)\cdots H(t_n)\}$$

Then the S-matrix is defined as the operator (assumed to be unitary)

$$S = \lim_{s\to-\infty,\, t\to\infty} \mathrm{e}^{iH_0 t}U(t,s)\mathrm{e}^{-iH_0 s}$$

(where it is assumed that the limits exist). If ψ_a and ψ'_b are normalized state vectors, what is the interpretation of

$$|(\psi'_b, S\psi_a)|^2 \;?$$

Hint: It may help to consider special cases first, e.g. that ψ_a and ψ'_b are eigenvectors of H_0.

(d) Let $H_I(t) = \mathrm{e}^{iH_0 t}H_1(t)\mathrm{e}^{-iH_0 t}$ and define

$$U_I(t,s) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_s^t dt_1 \int_s^t \dots \int_s^t dt_n\, T\{H_I(t_1)\cdots H_I(t_n)\}\,.$$

This is often called the dynamics in the *interaction picture*. Show that

$$S = \lim_{s\to-\infty,\, t\to\infty} U_I(t,s)\,.$$