Problem 5.1
Suppose that a classical scalar field \( \varphi(x) \) on Minkowski spacetime has a canonically conjugate momentum field \( \pi(x) \) and a Hamiltonian density \( H(\varphi(x), \pi(x)) \). Writing \( \varphi_t(x) = \varphi(t, x) \), \( \pi_t(x) = \pi(t, x) \), define \( h(\varphi_t, \pi_t) = \int d^3 x H(\varphi_t(x), \pi_t(x)) \) and define \( \delta H = \delta H(\varphi_t(x), \pi_t(x)) \) by the condition that
\[
\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} h(\varphi_t + \varepsilon \xi_t, \pi_t) = \int d^3 x \xi_t(x) \frac{\delta H}{\delta \varphi}(\varphi_t(x), \pi_t(x))
\]
for all variations \( \xi_t(x) \) in \( S(\mathbb{R}^4) \) of the field configurations. Define \( \delta H = \delta H(\varphi_t(x), \pi_t(x)) \) analogously.

Let the Hamiltonian density be given by
\[
H(\varphi_t(x), \pi_t(x)) = \frac{1}{2} \left( \pi_t(x)^2 + |\nabla \varphi_t(x)|^2 + m^2 \varphi_t(x)^2 \right) + \frac{\lambda}{p} \varphi_t(x)^p
\]
Show that the “Hamiltonian equations”
\[
\dot{\varphi}_t(x) = \frac{\delta H}{\delta \pi}(\varphi_t(x), \pi_t(x)), \quad \dot{\pi}_t(x) = -\frac{\delta H}{\delta \varphi}(\varphi_t(x), \pi_t(x))
\]
imply the field equation
\[
(\Box + m^2)\varphi(t, x) + \lambda \varphi(t, x)^{p-1} = 0.
\]

Problem 5.2
Let \( \phi(t, x) \) be the field operators of the interaction-free quantized Klein-Gordon field and let \( \pi(t, x) = \phi(t, x) \) the quantized canonically conjugate momentum field. These quantized field operators fulfill the equal-time commutation relations
\[
[\phi(t, x), \phi(t, y)] = 0 = [\pi(t, x), \pi(t, y)], \quad [\phi(t, x), \pi(t, y)] = i \delta(x - y) \cdot 1.
\]
Let \( H_0 \) be the Hamilton operator of the free quantized Klein-Gordon field, so that \( \dot{\phi}(t, x) = i[H_0, \phi(t, x)] \) and \( \dot{\pi}(t, x) = i[H_0, \pi(t, x)] \).

Suppose that a quantized Klein-Gordon field with self-interaction has field operators \( \Phi(t, x) \) and canonically conjugate momentum field operators \( \Pi(t, x) \) and a Hamilton operator \( H \) so that \( \dot{\Phi}(t, x) = i[H, \Phi(t, x)] \) and \( \dot{\Pi}(t, x) = i[H, \Pi(t, x)] \). Furthermore, suppose that for any time \( t \) there is a unitary operator \( U(t) \) (suitably differentiable in \( t \)) so that
\[
\Phi(t, x) = U(t)^{-1} \phi(t, x) U(t) \quad \text{and} \quad \Pi(t, x) = U(t)^{-1} \pi(t, x) U(t).
\]
(a) Show that the operators $\Phi(t, x)$ and $\Pi(t, x)$ fulfill the same equal-time commutation relations as $\phi(t, x)$ and $\pi(t, x)$.

(b) Show that there is a number-valued function $E(t)$ so that

$$i\dot{U}(t)U(t)^{-1} = U(t)H(t)U(t)^{-1} - H_0 + E(t) \cdot 1.$$ 

*Hint:* You can use that any operator $A$ that commutes for given $t$ with all $\phi(t, x)$ and all $\pi(t, x)$ is a multiple of the unit operator.

(c) Can $E(t)$ be chosen to be equal to 0 by a suitable re-definition of $U(t)$?

**Problem 5.3**

In a Hilbert space $\mathcal{H}$, let $H(t) = H_0 + H_1(t)$ be a time-dependent family of Hamilton operators.

(a) Show that a family of Hilbert space vectors $\psi_t, t \in \mathbb{R}$ is a solution to

$$i\frac{d}{dt}\psi_t = H(t)\psi_t, \quad \psi_s = \chi$$

for any given Hilbert space vector $\chi$ if

$$\psi_t = \left(1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_s^t dt_1 \int_s^{t_1} ... \int_s^{t_{n-1}} dt_n H(t_1) \cdots H(t_n)\right) \chi$$

(show this formally, assume the infinite sum converges).

(b) Show that the previous series formula (Dyson series) can be re-written in the form

$$\psi_t = \left(\sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_s^t dt_1 \int_s^{t_1} ... \int_s^{t_{n-1}} dt_n T\{H(t_1) \cdots H(t_n)\}\right) \chi$$

where $T\{H(t_1) \cdots H(t_n)\}$ is the time-ordered product of $H(t_1) \cdots H(t_n)$.

(c) Let

$$U(t, s) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_s^t dt_1 \int_s^{t_1} ... \int_s^{t_{n-1}} dt_n T\{H_I(t_1) \cdots H_I(t_n)\}$$

Then the $S$-matrix is defined as the operator (assumed to be unitary)

$$S = \lim_{s \to -\infty, t \to \infty} e^{iH_0t}U(t, s)e^{-iH_0s}$$

(where it is assumed that the limits exist). If $\psi_a$ and $\psi'_b$ are normalized state vectors, what is the interpretation of $|\langle \psi'_b, S \psi_a \rangle|^2$?

*Hint:* It may help to consider special cases first, e.g. that $\psi_a$ and $\psi'_b$ are eigenvectors of $H_0$.

(d) Let $H_I(t) = e^{iH_0t}H_1(t)e^{-iH_0t}$ and define

$$U_I(t, s) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_s^t dt_1 \int_s^{t_1} ... \int_s^{t_{n-1}} dt_n T\{H_I(t_1) \cdots H_I(t_n)\}.$$ 

This is often called the dynamics in the interaction picture. Show that

$$S = \lim_{s \to -\infty, t \to \infty} U_I(t, s).$$