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Relativistic Quantum Field Theory — Problem Sheet 4

2 pages — Problems 4.1 to 4.3

Problem 4.1 [In this problem, the metric sign convention opposite to Srednicki's is used]
Show that $SL(2,\mathbb{C})$ is the universal covering group of the proper orthochronous Lorentz group $\mathscr{L}_+^\uparrow$, with the covering map $\Lambda(\,.\,) : SL(2,\mathbb{C}) \to \mathscr{L}_+^\uparrow$ given by

$$\Lambda_{\mu\nu}(A) = \frac{1}{2}\mathrm{Tr}(A\sigma_\mu A^* \sigma_\nu)$$

where $\sigma_0 = \mathbf{1}$ is the 2×2 unit matrix and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices (see Problem 4.2 below).

For the proof, proceed along the following steps:

(1) Show that there is a one-to-one correspondence between coordinate vectors $x = (x^\mu)_{\mu=0,\dots,3}$ in Minkowski spacetime and hermitean 2×2 matrices H_x given by

$$H_x = x^\mu \sigma_\mu \,, \quad x^\mu = \eta^{\mu\nu}\mathrm{Tr}(H_x \sigma_\nu)$$

where $(\eta^{\mu\nu}) = (\eta_{\mu\nu}) = \mathrm{diag}(1,-1,-1,-1)$ and the **Einstein summation is employed**, i.e. doubly appearing indices (one of them downstairs, the other upstairs) are summed over.

(2) Show that

$$\det(H_x) = \eta_{\mu\nu}x^\mu x^\nu \,, \quad \frac{1}{2}(\det(H_x + H_y) - \det(H_x) - \det(H_y)) = \eta_{\mu\nu}x^\mu y^\nu \,.$$

(The 2nd equation results from the first by applying the parallellogram identity to symmetric bilinear forms such as the Minowski product $\eta(x,y) = \eta_{\mu\nu}x^\mu y^\nu$.)

(3) Use the previous findings to show that for any $A \in SL(2,\mathbb{C})$ there is some proper, orthochronous Lorentz transformation $\Lambda(A)$ such that

$$AH_x A^* = H_{\Lambda(A)x} \,.$$

(4) Show that $\Lambda(A)\Lambda(B) = \Lambda(AB)$, $\Lambda(\mathbf{1}_{2\times 2}) = \mathbf{1}_{4\times 4}$, $\Lambda(A) = \Lambda(B) \Rightarrow A = \pm B$, and that the matrix $\Lambda(A)$ is given by the equation above.

You may use the fact that $SL(2,\mathbb{C})$ is simply connected to conclude that (i) $\mathscr{L}_+^\uparrow$ is not simply connected and (ii) $SL(2,\mathbb{C})$ is the universal covering group of $\mathscr{L}_+^\uparrow$. If you like, you can also show that $SL(2,\mathbb{C})$ is simply connected.

Problem 4.2 [There was an error in a previously published version of this problem]
On $\mathcal{S}(\mathbb{R}^4, \mathbb{C}^2)$, the Schwartz functions on $\mathbb{R}^4$ with values in $\mathbb{C}^2$, introduce for given $m > 0$ the form

$$(\tilde{\varphi}, \tilde{\phi})_{1,m} = \int d^4p \, \delta(p_\mu p^\mu + m^2)\theta(p^0) \cdot \prec\tilde{\varphi}(p), \frac{-1}{m} p^\mu \sigma_\mu \tilde{\phi}(p)\succ$$

where $\prec x, y\succ = \overline{x}_1 y_1 + \overline{x}_2 y_2$ for $x = (x_1, x_2)^T$, $y = (y_1, y_2)^T$ in $\mathbb{C}^2$. The σ_μ are the Pauli-matrices,

$$\sigma_0 = \mathbf{1}_{\mathbb{C}^2}\,, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that $(\tilde{\varphi}, \tilde{\phi})_{1,m}$ has the properties of a scalar product, apart from positive-definedness; however, $(\tilde{\varphi}, \tilde{\varphi})_{1,m} \geq 0$ holds (but $(\tilde{\varphi}, \tilde{\varphi})_{1,m} = 0$ can occur for $\tilde{\varphi} \neq 0$).

(b) Show that $U_{(A,a)}$ defined by

$$U_{(A,a)}\tilde{\varphi}(p) = \mathrm{e}^{ip_\mu a^\mu} A\tilde{\varphi}(\Lambda(A^{-1})p)$$

is a (continuous) unitary representation of $\widetilde{\mathscr{P}}_+^\uparrow$, the universal covering group of the proper, orthochronous Poincaré group, on the Hilbert space $\mathfrak{h}_{1,m}$ obtained as completion of $\mathcal{S}(\mathbb{R}^4, \mathbb{C}^2)$ with respect to the scalar product $(\,.\,,\,.\,)_{1,m}$. (Strictly, one has to first divide out all $\tilde{\varphi}$ with $(\tilde{\varphi}, \tilde{\varphi})_{1,m} = 0$.)
If you are ready for a challenge, you can try to show that the representation is irreducible.

Problem 4.3
For $\tilde{\varphi}$ in $\mathcal{S}(\mathbb{R}^4, \mathbb{C}^2)$, define

$$\tilde{\chi}(p) = \frac{1}{m} p^\mu \sigma_\mu \tilde{\varphi}(p) \quad (p \in \mathbb{R}^4)$$

and define also

$$\varphi(x) = \int \frac{d^3\underline{p}}{2\omega_{\underline{p}}} \, \mathrm{e}^{-i(\underline{p}\cdot\underline{x} - \omega_{\underline{p}} x^0)} \tilde{\varphi}(\omega_{\underline{p}}, \underline{p})$$

where $\omega_{\underline{p}} = \sqrt{|\underline{p}|^2 + m^2}$, $\underline{p} \cdot \underline{x}$ is the Euclidean scalar product between 3-dimensional vectors $\underline{p}$ and $\underline{x}$, $x = (x^0, \underline{x}) \in \mathbb{R}^4$; $m > 0$ is a constant. The definition of $\chi(x)$ is analogous.

(a) Show that, with $\sigma^j = \sigma_j$ for $j = 1, 2, 3$,

$$i(\sigma_0\partial_0 + \sigma^j\partial_j)\varphi(x) = m\chi(x)\,, \quad i(\sigma_0\partial_0 - \sigma^j\partial_j)\chi(x) = m\varphi(x) \quad (x \in \mathbb{R}^4)\,.$$

(b) On writing

$$\psi(x) = \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

so that each $\psi(x)$ is an element in $\mathbb{C}^4$ and the γ^μ are complex 4×4 matrices, show that the equations for φ and χ in (a) can be written in the form

$$i\gamma^\mu \partial_\mu \psi(x) = m\psi(x) \quad (x \in \mathbb{R}^4)\,.$$

This is called the **Dirac equation**. The matrices γ^μ are called **Dirac matrices**.

(c) Show that

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{1}_{4\times4}$$