Problem 4.1  [In this problem, the metric sign convention opposite to Srednicki’s is used]

Show that $SL(2, \mathbb{C})$ is the universal covering group of the proper orthochronous Lorentz group $\mathcal{L}_+^\uparrow$, with the covering map $\Lambda(\cdot) : SL(2, \mathbb{C}) \to \mathcal{L}_+^\uparrow$ given by

$$\Lambda_{\mu\nu}(A) = \frac{1}{2} \text{Tr}(A \sigma_\mu A^* \sigma_\nu)$$

where $\sigma_0 = 1$ is the $2 \times 2$ unit matrix and $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices (see Problem 4.2 below).

For the proof, proceed along the following steps:

1. Show that there is a one-to-one correspondence between coordinate vectors $x = (x^\mu)_{\mu=0,...,3}$ in Minkowski spacetime and hermitean $2 \times 2$ matrices $H_x$ given by

$$H_x = x^\mu \sigma_\mu, \quad x^\mu = \eta^{\mu\nu} \text{Tr}(H_x \sigma_\nu)$$

where $(\eta^{\mu\nu}) = (\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1)$ and the Einstein summation is employed, i.e. doubly appearing indices (one of them downstairs, the other upstairs) are summed over.

2. Show that

$$\det(H_x) = \eta_{\mu\nu} x^\mu x^\nu, \quad \frac{1}{2} (\det(H_x + H_y) - \det(H_x) - \det(H_y)) = \eta_{\mu\nu} x^\mu y^\nu.$$  

(The 2nd equation results from the first by applying the parallelogram identity to symmetric bilinear forms such as the Minowski product $\eta(x, y) = \eta_{\mu\nu} x^\mu y^\nu$.)

3. Use the previous findings to show that for any $A \in SL(2, \mathbb{C})$ there is some proper, orthochronous Lorentz transformation $\Lambda(A)$ such that

$$AH_x A^* = H_{\Lambda(A)x}.$$  

4. Show that $\Lambda(A)\Lambda(B) = \Lambda(AB)$, $\Lambda(I_{2\times2}) = I_{4\times4}$, $\Lambda(A) = \Lambda(B) \Rightarrow A = \pm B$, and that the matrix $\Lambda(A)$ is given by the equation above.

You may use the fact that $SL(2, \mathbb{C})$ is simply connected to conclude that (i) $\mathcal{L}_+^\uparrow$ is not simply connected and (ii) $SL(2, \mathbb{C})$ is the universal covering group of $\mathcal{L}_+^\uparrow$. If you like, you can also show that $SL(2, \mathbb{C})$ is simply connected.
Problem 4.2 [There was an error in a previously published version of this problem]
On $S(\mathbb{R}^4, \mathbb{C}^2)$, the Schwartz functions on $\mathbb{R}^4$ with values in $\mathbb{C}^2$, introduce for given $m > 0$ the form
\[
(\tilde{\varphi}, \tilde{\psi})_{1,m} = \int d^4p \delta(p_\mu p^\mu + m^2)\theta(p^0) \cdot \langle \tilde{\varphi}(p), \frac{-1}{m}p^\mu \sigma_\mu \tilde{\psi}(p) \rangle
\]
where $\langle x, y \rangle = \mathcal{I}_{1}y_1 + \mathcal{I}_{2}y_2$ for $x = (x_1, x_2)^T$, $y = (y_1, y_2)^T$ in $\mathbb{C}^2$. The $\sigma_\mu$ are the Pauli-matrices,
\[
\sigma_0 = 1_{\mathbb{C}^2}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

(a) Show that $(\tilde{\varphi}, \tilde{\psi})_{1,m}$ has the properties of a scalar product, apart from positive-definedness; however, $(\tilde{\varphi}, \tilde{\varphi})_{1,m} \geq 0$ holds (but $(\tilde{\varphi}, \tilde{\varphi})_{1,m} = 0$ can occur for $\tilde{\varphi} \neq 0$).

(b) Show that $U_{(A,a)}$ defined by
\[
U_{(A,a)} \tilde{\varphi}(p) = e^{ip_\mu a^\mu} A \tilde{\varphi}(\Lambda(A^{-1})p)
\]
is a (continuous) unitary representation of $\tilde{\mathcal{P}}_4$, the universal covering group of the proper, orthochronous Poincaré group, on the Hilbert space $\mathcal{H}_{1,m}$ obtained as completion of $S(\mathbb{R}^4, \mathbb{C}^2)$ with respect to the scalar product $(\cdot, \cdot)_{1,m}$. (Strictly, one has to first divide out all $\tilde{\varphi}$ with $(\tilde{\varphi}, \tilde{\varphi})_{1,m} = 0$.)

If you are ready for a challenge, you can try to show that the representation is irreducible.

Problem 4.3
For $\tilde{\varphi}$ in $S(\mathbb{R}^4, \mathbb{C}^2)$, define
\[
\tilde{\chi}(p) = \frac{1}{m}p^\mu \sigma_\mu \tilde{\varphi}(p) \quad (p \in \mathbb{R}^4)
\]
and define also
\[
\varphi(x) = \int \frac{d^3p}{2\omega_p} e^{-i[p \cdot \bar{x} - \omega_p p^0]} \tilde{\varphi}(\omega_p, p)
\]
where $\omega_p = \sqrt{|p|^2 + m^2}$, $p \cdot \bar{x}$ is the Euclidean scalar product between 3-dimensional vectors $p$ and $\bar{x}$, $x = (x^0, \bar{x}) \in \mathbb{R}^4$; $m > 0$ is a constant. The definition of $\chi(x)$ is analogous.

(a) Show that, with $\sigma^j = \sigma_j$ for $j = 1, 2, 3$,
\[
i(\sigma_0 \partial_0 + \sigma^j \partial_j) \varphi(x) = m \chi(x), \quad i(\sigma_0 \partial_0 - \sigma^j \partial_j) \chi(x) = m \varphi(x) \quad (x \in \mathbb{R}^4).
\]

(b) On writing
\[
\psi(x) = \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}
\]
so that each $\psi(x)$ is an element in $\mathbb{C}^4$ and the $\gamma^\mu$ are complex $4 \times 4$ matrices, show that the equations for $\varphi$ and $\chi$ in (a) can be written in the form
\[
i\gamma^\mu \partial_\mu \psi(x) = m \psi(x) \quad (x \in \mathbb{R}^4).
\]
This is called the Dirac equation. The matrices $\gamma^\mu$ are called Dirac matrices.

(c) Show that
\[
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}1_{4 \times 4}
\]