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Relativistic Quantum Field Theory — Problem Sheet 3

2 pages — Problems 3.1 to 3.3

Problem 3.1
Let $\phi(x)$ denote the field operators of the quantized Klein-Gordon field.

(a) Show that the normal ordering of the product of two field operators can also be obtained by subtracting the vacuum expectation value from the operator product:

$$:\phi(x_1)\phi(x_2): = \phi(x_1)\phi(x_2) - \langle 0|\phi(x_1)\phi(x_2)|0\rangle \mathbf{1} = \phi(x_1)\phi(x_2) - w_2(x_1,x_2)\mathbf{1}$$

(b) Show that for any *odd* n, the vacuum expectation values $w_n(x_1,\dots,x_n) = \langle 0|\phi(x_1)\cdots\phi(x_n)|0\rangle$ vanish, $w_n(x_1,\dots,x_n) = 0$.

(c) Show explicitly that the 4-point Wightman function $w_4(x_1,x_2,x_3,x_4)$ can be written as a sum of products of 2-point functions.

Problem 3.2
Again, let $\phi(x)$ denote the field operators of the quantized Klein-Gordon field, and let $w_2(x_1,x_2) = \langle 0|\phi(x_1)\phi(x_2)|0\rangle$ be the Wightman 2-point function.

(a) Show that for all $f, g \in \mathcal{S}(\mathbb{R}^4)$,

$$\langle 0|\phi(f)^*\phi(g)|0\rangle = (\chi_f,\chi_g)_{\mathcal{K}} = \frac{1}{(2\pi)^3}\int \bar{\tilde{f}}(-\omega_{\underline{k}},-\underline{k})\tilde{g}(\omega_k,\underline{k})\,\frac{d^3k}{2\omega_{\underline{k}}}$$

where $\mathcal{K} = L^2(\mathbb{R}^3, d^3k/2\omega_{\underline{k}})$, $\chi_f(\underline{k}) = (2\pi)^{-3/2}\tilde{f}(\omega_{\underline{k}},\underline{k})$ and $\omega_{\underline{k}} = \sqrt{|\underline{k}|^2+m^2}$ ($\underline{k}\in\mathbb{R}^3$).[1]

(b) Show that the 2-point function can also be written as

$$w_2(x_1,x_2) = \frac{1}{(2\pi)^3}\int d^4k\, \mathrm{e}^{ik_\mu(x_1-x_2)^\mu}\,\delta(k_\mu k^\mu + m^2)\theta(k^0)$$

where θ is the Heaviside function, i.e. $\theta(s) = 1$ if $s \geq 0$ and $\theta(s) = 0$ if $s < 0$.

Hint: The rule of the game is, as usual, to first test-function integrate with respect to x_1 and x_2 and then to integrate over k, resp. evaluate the δ-distribution. In order to do that, first replace δ by a sequence h_j which approaches δ in the limit $j\to\infty$, $\lim_{j\to\infty}\int h_j(s)\varrho(s)\,ds = \varrho(0)$ for any test-function ϱ. Then for any $\eta > 0$, obtain a formula for $\lim_{j\to\infty}\int h_j(\eta - s^2)\varrho(s)\,ds$ by a suitable transformation of the integration variable.

[1] The definition of χ_f has been changed here compared to the previous definition in the lectures by the prefactor $(2\pi)^{-3/2}$ as this matches better with the conventions of Srednicki and other books.

(c) Show that the 2-point function $w_2(x_1, x_2)$ is a solution to the Klein-Gordon equation in both entries, i.e.

$$(\Box_{x_1} + m^2)w_2(x_1, x_2) = 0 = (\Box_{x_2} + m^2)w_2(x_1, x_2)$$

Problem 3.3

As before, $\phi(x)$ denote the field operators of the quantized Klein-Gordon field.
Define the *Feynman propagator* by

$$\Delta(x_1, x_2) \equiv \Delta(x_1 - x_1) = \lim_{\epsilon \to 0^+} \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{e}^{ik_\mu (x_1 - x_2)^\mu}}{k_\mu k^\mu + m^2 - i\epsilon}$$

(a) Show that

$$(\Box_{x_1} + m^2)\Delta(x_1 - x_2) = \delta(x_1 - x_2) = (\Box_{x_2} + m^2)\Delta(x_1 - x_2)$$

(b) Show that

$$\frac{1}{i}\Delta(x_1 - x_2) = \tau_2(x_1, x_2) = \langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle$$

To do this, show that (writing $x = (x^0, \underline{x})$ etc for four-vectors)

$$\Delta(x_1 - x_2) = \frac{i}{(2\pi)^3} \int \frac{d^3k}{2\omega_{\underline{k}}} \mathrm{e}^{i(\underline{k}\cdot(\underline{x}_1 - \underline{x}_2) - |x_1^0 - x_2^0|)}$$

which is best done by re-writing the expression for $\Delta(x_1 - x_2)$ as a contour integral in the complex k^0 plane and evaluating it using the residue theorem. Then show that the last expression equals $i\tau_2(x_1, x_2) = i\theta(x_1^0 - x_2^0)w_2(x_1, x_2) + i\theta(x_2^0 - x_1^0)w_2(x_2, x_1)$.