Problem 3.1
Let $\phi(x)$ denote the field operators of the quantized Klein-Gordon field.

(a) Show that the normal ordering of the product of two field operators can also be obtained by subtracting the vacuum expectation value from the operator product:

$$:\phi(x_1)\phi(x_2): = \phi(x_1)\phi(x_2) - \langle 0 | \phi(x_1)\phi(x_2) | 0 \rangle$$

(b) Show that for any odd $n$, the vacuum expectation values $w_n(x_1, \ldots, x_n) = \langle 0 | \phi(x_1) \cdots \phi(x_n) | 0 \rangle$ vanish, $w_n(x_1, \ldots, x_n) = 0$.

(c) Show explicitly that the 4-point Wightman function $w_4(x_1, x_2, x_3, x_4)$ can be written as a sum of products of 2-point functions.

Problem 3.2
Again, let $\phi(x)$ denote the field operators of the quantized Klein-Gordon field, and let $w_2(x_1, x_2) = \langle 0 | \phi(x_1)\phi(x_2) | 0 \rangle$ be the Wightman 2-point function.

(a) Show that for all $f, g \in S(\mathbb{R}^4)$,

$$\langle 0 | \phi(f)^* \phi(g) | 0 \rangle = (\chi_f, \chi_g)_\mathcal{K} = \frac{1}{(2\pi)^3} \int \tilde{f}(-\omega_k, -k) \tilde{g}(\omega_k, k) \frac{d^3k}{2\omega_k}$$

where $\mathcal{K} = L^2(\mathbb{R}^3, d^3k/2\omega_k)$, $\chi_f(k) = (2\pi)^{-3/2} \tilde{f}(\omega_k, k)$ and $\omega_k = \sqrt{|k|^2 + m^2}$ ($k \in \mathbb{R}^3$).

(b) Show that the 2-point function can also be written as

$$w_2(x_1, x_2) = \frac{1}{(2\pi)^3} \int d^4k e^{ik \cdot (x_1 - x_2)} \delta(k_\mu k^\mu + m^2) \theta(k^0)$$

where $\theta$ is the Heaviside function, i.e. $\theta(s) = 1$ if $s \geq 0$ and $\theta(s) = 0$ if $s < 0$.

Hint: The rule of the game is, as usual, to first test-function integrate with respect to $x_1$ and $x_2$ and then to integrate over $k$, resp. evaluate the $\delta$-distribution. In order to do that, first replace $\delta$ by a sequence $h_j$ which approaches $\delta$ in the limit $j \to \infty$, $\lim_{j \to \infty} \int h_j(s) g(s) ds = g(0)$ for any test-function $g$. Then for any $\eta > 0$, obtain a formula for $\lim_{j \to \infty} \int h_j(\eta - s^2) g(s) ds$ by a suitable transformation of the integration variable.

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1The definition of $\chi_f$ has been changed here compared to the previous definition in the lectures by the prefactor $(2\pi)^{-3/2}$ as this matches better with the conventions of Srednicki and other books.
(c) Show that the 2-point function \( w_2(x_1, x_2) \) is a solution to the Klein-Gordon equation in both entries, i.e.

\[
(\Box + m^2) w_2(x_1, x_2) = 0 = (\Box + m^2) w_2(x_1, x_2)
\]

**Problem 3.3**

As before, \( \phi(x) \) denote the field operators of the quantized Klein-Gordon field.

Define the **Feynman propagator** by

\[
\Delta(x_1, x_2) \equiv \Delta(x_1 - x_2) = \lim_{\epsilon \to 0^+} \int \frac{d^4k}{(2\pi)^4} \frac{e^{i k_\mu (x_1 - x_2)\mu}}{k_\mu k^\mu + m^2 - i\epsilon}
\]

(a) Show that

\[
(\Box + m^2) \Delta(x_1 - x_2) = \delta(x_1 - x_2) = (\Box + m^2) \Delta(x_1 - x_2)
\]

(b) Show that

\[
\frac{1}{i} \Delta(x_1 - x_2) = \tau_2(x_1, x_2) = \langle 0 | T\{\phi(x_1)\phi(x_2)\} | 0 \rangle
\]

To do this, show that (writing \( x = (x^0, \vec{x}) \) etc for four-vectors)

\[
\Delta(x_1 - x_2) = \frac{i}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} e^{i\vec{k} \cdot (x_1 - x_2) - |x^0_1 - x^0_2|}
\]

which is best done by re-writing the expression for \( \Delta(x_1 - x_2) \) as a contour integral in the complex \( k^0 \) plane and evaluating it using the residue theorem. Then show that the last expression equals \( i\tau_2(x_1, x_2) = i\theta(x^0_1 - x^0_2)w_2(x_1, x_2) + i\theta(x^0_2 - x^0_1)w_2(x_2, x_1) \).