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Relativistic Quantum Field Theory — Problem Sheet 2

3 pages — Problems 2.1 to 2.3

Problem 2.1

For the quantized Klein-Gordon field with L-periodicity in all space coordinates has been defined (formally) as

$$\phi(t,\underline{x}) = \frac{1}{L^{3/2}} \sum_{\underline{\ell}} \frac{1}{2\omega_{\underline{k}}} \left(\mathrm{e}^{i(\underline{\ell}\cdot\underline{x} - \omega_{\underline{\ell}} t)} a_{\underline{\ell}} + \text{h.c.} \right)$$

with the annihilation and creation operators $a_{\underline{k}}$, $a_{\underline{\ell}}^{+}$ fulfilling the commutation relations

$$[a_{\underline{k}}, a_{\underline{\ell}}^{+}] = \delta_{\underline{k}\,\underline{\ell}} \cdot \mathbf{1}\,, \quad [a_{\underline{k}}, a_{\underline{\ell}}] = 0 = [a_{\underline{k}}^{+}, a_{\underline{\ell}}^{+}]\,, \quad a_{\underline{k}}^{+} = a_{\underline{k}}^{*}$$

The renormalized Hamilton operator has been defined (formally) as

$$\mathbb{H} = \sum_{\underline{k}} \omega_{\underline{k}} a_{\underline{k}}^{+} a_{\underline{k}}$$

where $\omega_{\underline{k}} = \sqrt{\underline{k}\cdot\underline{k} + m^2}\,.$

Show that the following holds (formally):

1. $[\mathbb{H}, \phi(t,\underline{x})] = i\partial_t \phi(t,\underline{x})$
2. $(\Box + m^2)\phi(t,\underline{x}) = 0$

Writing $x = (x^\mu)$, $t = x^0$, $\underline{x} = (x^1, x^2, x^3)$ and $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$, so that 'primed' and 'unprimed' spacetime coordinates are related by a Poincaré transformation, define

$$\phi'(x') = \phi(x)$$

and show that (formally)

$$(\Box' + m^2)\phi'(x') = 0$$

Problem 2.2

Prologue: Bose-Fock space

Let $\mathcal{H} = L^2(\mathbb{R}^n, \sigma d^n y)$ be a Hilbert space given as L^2 space of functions on $\mathbb{R}^n$ with respect to the Lebesgue measure with a density σ, so that the scalar product on $\mathcal{H}$ is given by

$$(\chi, \psi) = \int_{\mathbb{R}^n} \overline{\chi(y)} \psi(y)\, \sigma(y) d^n y$$

The N-fold tensor product $\otimes^N \mathcal{H}$ of $\mathcal{H}$ is then the space of functions κ on $\mathbb{R}^{nN}$ so that

$$\int |\kappa(y_{(1)}, \ldots, y_{(N)})|^2 \sigma(y_{(1)}) d^n y_{(1)} \cdots \sigma(y_{(N)}) d^n y_{(N)} < \infty$$

Thus, $\otimes^N \mathcal{H}$ is also an L^2 space with scalar product

$$(\kappa, \varrho)_N = \int \overline{\kappa(y_{(1)}, \ldots, y_{(N)})} \varrho(y_{(1)}, \ldots, y_{(N)}) \sigma(y_{(1)}) d^n y_{(1)} \cdots \sigma(y_{(N)}) d^n y_{(N)}$$

Given $\chi_j \in \mathcal{H}$, $j = 1, \ldots, N$, their tensor product

$$\chi_1 \otimes \cdots \otimes \chi_N(y_{(1)}, \ldots, y_{(N)}) = \chi_1(y_{(1)}) \cdots \chi_N(y_{(N)})$$

is a function in $\otimes^N \mathcal{H}$. The set of linear combinations of such tensor products is a dense subspace in $\otimes^N \mathcal{H}$. The symmetrization operator on $\otimes^N \mathcal{H}$ given by

$$P_+ \kappa(y_{(1)}, \ldots, y_{(N)}) = \frac{1}{N!} \sum_{\pi} \kappa(y_{\pi(1)}, \ldots, y_{\pi(N)}),$$

where the sum runs over all permutations π of the N indices, turns any $\kappa \in \otimes^N \mathcal{H}$ into a function that is symmetric under any permutation of its arguments. P_+ can be shown to be a projector in $\otimes^N \mathcal{H}$ and we write $\mathcal{H}_+^N = P_+(\otimes^N \mathcal{H})$ ("symmetric N-fold product of $\mathcal{H}$"). The *symmetric Fock space* (or *Bose-Fock space*) $\mathcal{F}_+(\mathcal{H})$ *over* $\mathcal{H}$ is then defined as the Hilbert space of all sequences $\boldsymbol{\kappa} = (\kappa_N)_{N=0}^{\infty}$ with $\kappa_0 \in \mathbb{C}$ and $\kappa_N \in \mathcal{H}_+^N$ for $N \geq 1$, and with the property

$$(\boldsymbol{\kappa}, \boldsymbol{\kappa})_{\mathcal{F}} < \infty$$

where the scalar product on $\mathcal{F}_+(\mathcal{H})$ is given by

$$(\boldsymbol{\kappa}, \boldsymbol{\psi})_{\mathcal{F}} = \sum_{N=0}^{\infty} (\kappa_N, \psi_N)_N$$

A special vector in $\mathcal{F}_+(\mathcal{H})$ is the *vacuum vector* $\boldsymbol{\psi}_0$ defined by $\psi_{0\,N} = \delta_{0\,N}$.

Given any $\chi \in \mathcal{H}$, one can define the *creation operator* $a^+(\chi)$ in $\mathcal{F}_+(\mathcal{H})$ which assigns to $\boldsymbol{\kappa} = (\kappa_N)_{N=0}^{\infty}$ the vector (or sequence) $a^+(\chi)\boldsymbol{\kappa}$ in the Bose-Fock space whose $N = 0$ entry is $= 0$ and whose $N \geq 1$ entry is

$$\sqrt{N} P_+(\chi \otimes \kappa_{N-1})$$

The *annihilation operator* $a(\chi)$ is the hermitean adjoint of χ and is given by setting $a(\chi)\boldsymbol{\psi}_0 = 0$ and by defining $a(\chi)\boldsymbol{\kappa}$ as the vector (or sequence) in the Bose-Fock space whose $N \geq 0$ entry is given by

$$\sqrt{N+1} \int \kappa_{N+1}(y_{(1)}, \ldots, y_{(N)}, y_{(N+1)}) \overline{\chi(y_{(N+1)})} \sigma(y_{(N+1)}) d^n y_{(N+1)}$$

Actually, these operators are *unbounded* and hence one must take care about the domain of definition. One usually considers as elements in the domain of definition those sequences in the Bose-Fock space so that only finitely many of the different N entries are different from 0.

After this long prologue:

1. How do $a(\chi)$ and $a^+(\chi)$ act on $\psi \otimes \psi \otimes \cdots \otimes \psi \in \mathcal{H}_+^M$ for some M in $\mathbb{N}$ — viewing this as a vector (or sequence) in the Bose-Fock space by setting all N entries other than $N = M$ equal to 0 ?

2. Show that $a^+(\chi)$ is linear in χ while $a(\chi)$ is conjugate-linear in χ

3. Show that $a^+(\chi)$ is the hermitean adjoint of $a(\chi)$

4. Show that $[a(\chi), a^+(\psi)] = (\chi, \psi) \cdot \mathbf{1}$ (scalar product in $\mathcal{H}$)

5. $[a(\chi), a(\psi)] = ?$ $\quad [a^+(\chi), a^+(\psi)] = ?$

6. Let U be a unitary operator on $\mathbb{H}$. An operator $\mathbb{U}$ on $\mathcal{F}_+(\mathcal{H})$ can be defined by setting

$$\mathbb{U}P_+(\chi_1 \otimes \cdots \otimes \chi_N) = P_+(U\chi_1 \otimes \cdots \otimes U\chi_N)$$

along with the convention in the first question. Show that $\mathbb{U}$ is a unitary operator on $\mathcal{F}_+(\mathcal{H})$ (the "second quantization" of U), and

$$\mathbb{U}a(\chi)\mathbb{U}^* = a(U\chi)\,, \quad \mathbb{U}a^+(\chi)\mathbb{U}^* = a^+(U\chi)$$

Problem 2.3

Define $\mathcal{K} = L^2(\mathbb{R}^3, \sigma d^3k)$ where $\sigma(\underline{k}) = 1/2\omega_{\underline{k}}$ with $\omega_{\underline{k}}$ defined as in problem 2.1. The formal creation and annihilation operators $a_{\underline{k}}^+$ and $a_{\underline{k}}$ can be regarded as (real-linear) operator-valued distributions in $\mathcal{F}_+(\mathcal{K}$ by identifying $a(f) = \int \overline{f(\underline{k})} a_{\underline{k}}\, d^3k$ and $a^+(f) = \int f(\underline{k}) a_{\underline{k}}^+\, d^3k$ for $f \in \mathcal{S}(\mathbb{R}^3)$.

(a) The quantized Klein-Gordon field operator is (formally) defined as

$$\phi(t, \underline{x}) = \int \frac{1}{2\omega_{\underline{k}}} \left(\mathrm{e}^{i(\underline{\ell}\cdot\underline{x} - \omega_{\underline{\ell}} t)} a_{\underline{\ell}} + \text{h.c.} \right) d^3k$$

For any $F \in \mathcal{S}(\mathbb{R}^4)$, define

$$\chi_F(\underline{k}) = \tilde{F}(\omega_{\underline{k}}, \underline{k})\,,$$

where $\tilde{F}(k) = \int \mathrm{e}^{-ik_\mu x^\mu} F(x)\, d^4x$ for $F \in \mathcal{S}(\mathbb{R}^4)$. Hence, χ_F is a vector in $\mathcal{K}$. Show that for any $F \in \mathcal{S}(\mathbb{R}^4)$

$$\begin{aligned} \phi(F) &= \int F(t, \underline{x}) \phi(t, \underline{x})\, dt\, d^3x \\ &= a(\chi_{\overline{F}}) + a^+(\chi_F) \end{aligned}$$

where $\overline{F}$ means complex conjugation of F.

(b) Writing $x = (x^\mu)$, $t = x^0$, $\underline{x} = (x^1, x^2, x^3)$, show that for any Lorentz transformation Λ there is a unitary operator $\mathbb{U}(\Lambda)$ on $\mathcal{F}_+(\mathcal{K})$ so that

$$\mathbb{U}(\Lambda)\phi(x)\mathbb{U}^*(\Lambda) = \phi(\Lambda x)$$

with $(\Lambda x)^\mu = \Lambda^\mu{}_\nu x^\nu$. Show also that $\mathbb{U}(\Lambda_1)\mathbb{U}(\Lambda_2) = \mathbb{U}(\Lambda_1\Lambda_2)$ (matrix product of Lorentz transformation matrices).