Problem 1.1
Let $x^\mu$ and $x'^\mu$ be coordinates with respect to two inertial systems related by a Poincaré transformation,

\[ x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu. \]

(a) The defining property of a Lorentz transformation $\Lambda^\mu_\nu$ is preservation of the light cone, and can be characterized as

\[ g_{\mu\nu} \Lambda^\mu_\rho \Lambda^\rho_\sigma = g_{\sigma\sigma}, \]

where $(g_{\mu\nu}) = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. Show that (**) is equivalent to the preservation of Lorentzian distance, i.e.

\[ g_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) = g_{\mu\nu}(x'^\mu - y'^\mu)(x'^\nu - y'^\nu) \quad \text{for all} \quad x = (x^\mu), \quad y = (y^\mu), \]

where $y'^\mu$ and $y^\mu$ are related as in (*) by the same Poincaré transformation as $x'^\mu$ and $x^\mu$.

(b) A Lorentz transformation $\Lambda^\mu_\nu$ is called orthochronous if it preserves the direction of time, i.e. if $x'^0 > 0$ whenever $x^0 > 0$. Show that $\Lambda^\mu_\nu$ is orthochronous if and only if $\Lambda^0_0 \geq 1$. Give an example of a Lorentz transformation that is antichronous, i.e. not orthochronous.

(c) Suppose that the coordinate axes $\vec{e}_j$ and $\vec{e}_j'$ coincide (and have the same base point) at $t = t' = 0$ and that the ‘primed’ inertial system is moving in direction $\vec{e}_1$ with respect to the ‘unprimed’ inertial system with constant velocity $v$. Then according to special relativity, one has

\[ x'^0 = \gamma \cdot (x^0 - (v/c)x^1), \quad x'^1 = \gamma \cdot (x^1 - (v/c)x^0), \quad x'^2 = x^2, \quad x'^3 = x^3 \]

where $\gamma = (1 - (v/c)^2)^{-1/2}$. Show that this corresponds to (*) with $a^\mu = 0$ and with a Lorentz transformation of the form

\[ (\Lambda^\mu_\nu) = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) & 0 & 0 \\ \sinh(\theta) & \cosh(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]
with a real number \( \theta = \theta(v/c) \) called the **rapidity of the Lorentz boost**. Determine \( \theta(v/c) \).

**Problem 1.2**

Again, let \( x^\mu \) and \( x'^\mu \) be coordinates with respect to two inertial systems related by a Poincaré transformation

\[
(*) \quad x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu.
\]

Let \( \psi(x) = \psi(x^0, x^1, x^2, x^3) \) be a smooth solution to the free Schrödinger equation with respect to the ‘unprimed’ inertial system, i.e.

\[
i\hbar c \frac{\partial}{\partial x^0} \psi(x) = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial (x^1)^2} + \frac{\partial^2}{\partial (x^2)^2} + \frac{\partial^2}{\partial (x^3)^2} \right) \psi(x)
\]

where \( m \) represents the mass of a free particle.

On setting \( \psi'(x') = \psi(x) \), one would say that (any solution to) the Schrödinger equation transforms covariantly with respect to Poincaré transformations if \( \psi'(x') \) is a solution to the Schrödinger equation with respect to the ‘primed’ inertial system. Show that this fails in general.

**Problem 1.3**

Once more, let \( x^\mu \) and \( x'^\mu \) be coordinates with respect to two inertial systems related by a Poincaré transformation

\[
(*) \quad x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu.
\]

Let \( \varphi(x) = \varphi(x^0, x^1, x^2, x^3) \) be a smooth solution to the Klein-Gordon equation with respect to the ‘unprimed’ inertial system,

\[
(\Box + m^2) \varphi(x) = 0
\]

where \( m \geq 0 \) represents a particle mass and

\[
\Box = -\partial^\mu \partial_\mu = -\left( -\frac{\partial^2}{\partial (x^0)^2} + \frac{\partial^2}{\partial (x^1)^2} + \frac{\partial^2}{\partial (x^2)^2} + \frac{\partial^2}{\partial (x^3)^2} \right)
\]

is the Klein-Gordon operator with respect to the ‘unprimed’ coordinates.

(a) Show that (any solution to) the Klein-Gordon equation transforms covariantly with respect to Poincaré transformations. That is, on setting \( \varphi'(x') = \varphi(x) \), the function \( \varphi'(x') \) is a solution to the Klein-Gordon equation with respect to the ‘primed’ inertial system.

(b) Find 4-dimensional wave vectors \( k = (k_\mu) \) so that, for any \( a \in \mathbb{C} \), the **plane wave**

\[
\phi(x) = ae^{ik_\mu x^\mu}
\]

is a solution to the Klein-Gordon equation. In other words, find the **dispersion relation** expressing \( k_0 \) as a function of \( k_1, k_2, k_3 \) so that the plane wave is a solution to the Klein-Gordon equation.
(c) Let \( \varrho(x) = |\varphi(x)|^2 \). Following the line of argument used in quantum mechanics, in order that \( \varrho(x) \) may be viewed as a probability density, it should fulfill a continuity equation of the form

\[
c c \frac{\partial}{\partial x^0} \varrho(x) + \sum_{k=1}^3 \frac{\partial}{\partial x^k} j^k(x) = 0
\]

with \( j^k(x) = \eta \text{Im}(\overline{\varphi(x)} \frac{\partial}{\partial x^k} \varphi(x)) \) where \( \eta \) is a suitable constant. Show that this does not hold in general for solutions \( \varphi(x) \) to the Klein-Gordon equation by giving a counterexample — see, e.g., item (b).

(d) Show that any smooth solution \( \varphi(x) \) to the Klein-Gordon equation fulfills

\[
\partial^\mu \left( \overline{\varphi(x)} \partial_\mu \varphi(x) - \varphi(x) \partial_\mu \overline{\varphi(x)} \right) = 0.
\]

This would suggest that \( \chi(x) = \overline{\varphi(x)} \partial_0 \varphi(x) - \varphi(x) \partial_0 \overline{\varphi(x)} \) might be a candidate for a probability density (at time \( x^0 \)) associated with \( \varphi \), as it satisfies a continuity equation. However, show that this is not tenable since, for any given \( x^0 \), there are always solutions to the Klein-Gordon equation so that \( \chi(x^0, x^1, x^2, x^3) \) can be positive or negative. To show this, you can make use of the well-posedness of the Cauchy-problem for the Klein-Gordon equation: Given any time \( x^0 \), and any pair of Schwartz functions \( u, v \in S(\mathbb{R}^3) \), there is a unique, smooth solution \( \varphi \) to the Klein-Gordon equation so that

\[
\varphi(x^0, \mathbf{x}) = u(\mathbf{x}) \quad \text{and} \quad \frac{\partial}{\partial s} \varphi(s, \mathbf{x}) \bigg|_{s=x^0} = v(\mathbf{x}) \quad (\mathbf{x} = (x^1, x^2, x^3) \in \mathbb{R}^3).
\]