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Relativistic Quantum Field Theory — Problem Sheet 1

3 pages — Problems 1.1 to 1.3

Problem 1.1

Let x^μ and x'^μ be coordinates with respect to two inertial systems related by a Poincaré transformation

$$(\star) \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \, .$$

(a) The defining property of a Lorentz transformation $\Lambda^\mu{}_\nu$ is preservation of the light cone, and can be characterized as

$$(\star\star) \qquad g_{\mu\nu}\Lambda^\mu{}_\varrho \Lambda^\nu{}_\sigma = g_{\varrho\sigma}$$

where $(g_{\mu\nu}) = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric. Show that $(\star\star)$ is equivalent to the *preservation of Lorentzian distance*, i.e.

$$g_{\mu\nu}(x^\mu - y^\mu)(x^\nu - y^\nu) = g_{\mu\nu}(x'^\mu - y'^\mu)(x'^\nu - y'^\nu) \quad \text{for all} \quad x = (x^\mu)\, ,\ y = (y^\mu)\, ,$$

where y'^μ and y^μ are related as in $(\star)$ by the same Poincaré transformation as x'^μ and x^μ.

(b) A Lorentz transformation $\Lambda^\mu{}_\nu$ is called *orthochronous* if it preserves the direction of time, i.e. if $x'^0 > 0$ whenever $x^0 > 0$. Show that $\Lambda^\mu{}_\nu$ is orthochronous if and only if $\Lambda^0{}_0 \geq 1$. Give an example of a Lorentz transformation that is *antichronous*, i.e. not orthochronous.

(c) Suppose that the coordinate axes $\vec{e}_j$ and $\vec{e}_j{}'$ coincide (and have the same base point) at $t = t' = 0$ and that the 'primed' inertial system is moving in direction $\vec{e}_1$ with respect to the 'unprimed' inertial system with constant velocity v. Then according to special relativity, one has

$$x'^0 = \gamma \cdot (x^0 - (v/c)x^1)\, , \quad x'^1 = \gamma \cdot (x^1 - (v/c)x^0)\, , \quad x'^2 = x^2\, , \quad x'^3 = x^3$$

where $\gamma = (1 - (v/c)^2)^{-1/2}$. Show that this corresponds to $(\star)$ with $a^\mu = 0$ and with a Lorentz transformation of the form

$$(\Lambda^\mu{}_\nu) = \begin{pmatrix} \cosh(\theta) & \sinh(\theta) & 0 & 0 \\ \sinh(\theta) & \cosh(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with a real number $\theta = \theta(v/c)$ called the *rapidity of the Lorentz boost.* Determine $\theta(v/c)$.

Problem 1.2

Again, let x^μ and x'^μ be coordinates with respect to two inertial systems related by a Poincaré transformation

$$(\star) \qquad\qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \,.$$

Let $\psi(x) = \psi(x^0, x^1, x^2, x^3)$ be a smooth solution to the free Schrödinger equation with respect to the 'unprimed' inertial system, i.e.

$$i\hbar c \frac{\partial}{\partial x^0}\psi(x) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial (x^1)^2} + \frac{\partial^2}{\partial (x^2)^2} + \frac{\partial^2}{\partial (x^3)^2}\right)\psi(x)$$

where m represents the mass of a free particle.

On setting $\psi'(x') = \psi(x)$, one would say that (any solution to) the Schrödinger equation *transforms covariantly with respect to Poincaré transformations* if $\psi'(x')$ is a solution to the Schrödinger equation with respect to the 'primed' inertial system. Show that this fails in general.

Problem 1.3

Once more, let x^μ and x'^μ be coordinates with respect to two inertial systems related by a Poincaré transformation

$$(\star) \qquad\qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \,.$$

Let $\varphi(x) = \varphi(x^0, x^1, x^2, x^3)$ be a smooth solution to the Klein-Gordon equation with respect to the 'unprimed' inertial system,

$$(\Box + m^2)\varphi(x) = 0$$

where $m \geq 0$ represents a particle mass and

$$\Box = -\partial^\mu\partial_\mu = -\left(-\frac{\partial^2}{\partial (x^0)^2} + \frac{\partial^2}{\partial (x^1)^2} + \frac{\partial^2}{\partial (x^2)^2} + \frac{\partial^2}{\partial (x^3)^2}\right)$$

is the Klein-Gordon operator with respect to the 'unprimed' coordinates.

(a) Show that (any solution to) the Klein-Gordon equation transforms covariantly with respect to Poincaré transformations. That is, on setting $\varphi'(x') = \varphi(x)$, the function $\varphi'(x')$ is a solution to the Klein-Gordon equation with respect to the 'primed' inertial system.

(b) Find 4-dimensional wave vectors $k = (k_\mu)$ so that, for any $a \in \mathbb{C}$, the *plane wave*

$$\phi(x) = a\mathrm{e}^{ik_\mu x^\mu}$$

is a solution to the Klein-Gordon equation. In other words, find the *dispersion relation* expressing k_0 as a function of k_1, k_2, k_3 so that the plane wave is a solution to the Klein-Gordon equation.

(c) Let $\varrho(x) = |\varphi(x)|^2$. Following the line of argument used in quantum mechanics, in order that $\varrho(x)$ may be viewed as a probability density, it should fulfill a continuity equation of the form

$$c\frac{\partial}{\partial x^0}\varrho(x) + \sum_{k=1}^{3}\frac{\partial}{\partial x^k}j^k(x) = 0$$

with $j^k(x) = \eta \mathrm{Im}(\overline{\varphi(x)}\frac{\partial}{\partial x^k}\varphi(x))$ where η is a suitable constant. Show that this *does not hold* in general for solutions $\varphi(x)$ to the Klein-Gordon equation by giving a counterexample — see, e.g., item (b).

(d) Show that any smooth solution $\varphi(x)$ to the Klein-Gordon equation fulfills

$$\partial^\mu\left(\overline{\varphi(x)}\partial_\mu\varphi(x) - \varphi(x)\partial_\mu\overline{\varphi(x)}\right) = 0\,.$$

This would suggest that $\chi(x) = \overline{\varphi(x)}\partial_0\varphi(x) - \varphi(x)\partial_0\overline{\varphi(x)}$ might be a candidate for a probability density (at time x^0) associated with φ, as it satisfies a continuity equation. However, show that this is not tenable since, for any given x^0, there are always solutions to the Klein-Gordon equation so that $\chi(x^0, x^1, x^2, x^3)$ can be positive or negative. To show this, you can make use of the *well-posedness of the Cauchy-problem* for the Klein-Gordon equation: Given any time x^0, and any pair of Schwartz functions $u, v \in \mathcal{S}(\mathbb{R}^3)$, there is a unique, smooth solution φ to the Klein-Gordon equation so that

$$\varphi(x^0, \underline{x}) = u(\underline{x}) \quad \text{and} \quad \left.\frac{\partial}{\partial s}\varphi(s, \underline{x})\right|_{s=x^0} = v(\underline{x}) \quad (\underline{x} = (x^1, x^2, x^3) \in \mathbb{R}^3)\,.$$