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Advanced Statistical Physics - Problem Set 12

Summer Term 2025

Due Date: Thursday, June 26, 17:00. Hand in tasks marked with * via Moodle.

*1. Specific heat exponent and scaling relation *4+4 Points*

(a) Calculate the specific heat critical exponent using

$$C_{\text{sing}}(t,h) = -T\frac{\partial^2}{\partial T^2} f_{\text{sing}}(t,h) \ ,$$

and the scaling hypothesis for $f_{\text{sing}}(t,h)$. Start from the generalized homogeneity equation

$$\lambda f_{\text{sing}}(t,h) = f_{\text{sing}}(\lambda^{a_t} t, \lambda^{a_h} h) \ .$$

Hint: Use an appropriate expression for λ to obtain the form of the singular part of the free energy as given in the lectures

$$f_{\text{sing}}(t,h) = |t|^c g_{f,\pm}(h/|t|^{\Delta}) \ .$$

Use the scaling of C_{sing} to relate c and α.

(b) Exactly at the critical point, the correlation length is infinite, and therefore all correlations decay as a power-law of the separation. From scattering experiments one obtains that

$$\langle m(\boldsymbol{x}) m(0)\rangle_c \sim 1/|\boldsymbol{x}|^{d-2+\eta} \ .$$

Away from criticality, the correlation functions decay exponentially with the length scale determined by the correlation length $\xi(t,h)$. This exponential decay can be approximated with an abrupt decay of the correlation function to zero at $|\boldsymbol{x}| \sim \xi(t,h)$.

Use the definition of the susceptibility

$$\chi \sim \int d^d x \langle m(\boldsymbol{x}) m(0)\rangle_c$$

to derive Fisher's identity, which establishes a connection between the correlation length exponent ν, the correlation function exponent η, and the susceptibility exponent γ.

2. Hartree approximation *2+3+3+1+1+2 Points*

Consider the Landau-Ginzburg Hamiltonian:

$$\beta\mathcal{H} = \int d^d x \left[\frac{t}{2}\boldsymbol{m}^2 \ + \frac{K}{2}(\nabla \boldsymbol{m})^2 + u(\boldsymbol{m}^2)^2\right] \ ,$$

describing an N-component magnetization vector $\boldsymbol{m}(\boldsymbol{x})$. Assume that $t > 0$.

(a) Perform a Hubbard-Stratonovich trasnformation by first multiplying the partition function by

$$\mathbb{1} = \int D\rho(\boldsymbol{x})\ e^{-N^2 \int d^d x \rho(\boldsymbol{x})^2/2}$$

and performing a shift $\rho \to \rho + \alpha \boldsymbol{m}^2$, and show that with suitably chosen α you obtain a new Hamiltonian

$$\beta\mathcal{H}[m,\rho] = \int d^d x \left[\frac{t + 2N^2\alpha\rho}{2}\boldsymbol{m}^2 + \frac{K}{2}(\nabla \boldsymbol{m})^2 + \frac{N^2\rho^2}{2}\right] .$$

(b) We want to find a saddle-point equation, where $\rho(\boldsymbol{x}) = \rho_0$. Therefore, assume that ρ is constant in space and integrate over $\boldsymbol{m}$ so that you obtain an effective Hamiltonian for ρ_0

$$\beta\mathcal{H}_{\text{eff}}(\rho_0) = \frac{N^2\rho_0^2 V}{2} + \frac{N}{2}\sum_{\boldsymbol{q}} \ln\left(t + 2N^2\alpha\rho_0 + Kq^2\right) .$$

(c) Use the effective Hamiltonian obtained in (b) to find the saddle-point equation for ρ_0. Notice that in the Hamiltonian obtained in part (a), t has been renormalized so that $t' = t + 2N^2\alpha\rho$. Use the saddle-point equation to find the self-consistency equation for t'

$$t' = t + \frac{4uN}{(2\pi)^d}\int d^d q\, \frac{1}{t' + Kq^2} .$$

(d) Argue why the method used above works well in the limit $N \to \infty$.

(e) In order to determine the critical exponents, we need to know how the renormalized parameter t' depends on the reduced temperature $t - t_c$. It can be shown that by introducing a new variable $M^2 = t'/(t - t_c)$, the self-consistency equation for $2 < d < 4$ can be rewritten as

$$1 = M^2\left[1 + C(d)N\left(\frac{t_G}{M^2(t - t_c)}\right)^{\frac{4-d}{2}}\right] ,$$

where t_G is the Ginzburg temperature and $C(d)$ is a constant. Solve the equation in the limits $t - t_c \gg t_G$ and $t - t_c \ll t_G$.

(f) Comment on how the critical exponents for the susceptibility γ, correlation length ν, specific heat α, and magnetization β are changed as compared to the result of the Gaussian model in the two limits discussed in (e).