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Advanced Statistical Physics - Problem Set 3

Summer Term 2025

Due Date: Thursday, April 24, 17:00. Hand in tasks marked with * via Moodle.

1. Lindhard response function

2 + 5 + 5 + 2 + 2 Points

As derived in the lectures, the response of a d-dimensional free electron gas to a time-independent potential $\phi(\boldsymbol{r})$ is described by an induced charge $\varrho^{\text{ind}}(\boldsymbol{r})$ with Fourier transform

$$\varrho^{\text{ind}}(\boldsymbol{q}) = \Pi_0(\boldsymbol{q})\,\phi(\boldsymbol{q}) \ ,$$

where $\Pi_0(\boldsymbol{q}, \omega = 0) \equiv \Pi_0(\boldsymbol{q})$ is the Lindhard response function given by

$$\Pi_0(\boldsymbol{q}) = \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \, \frac{f_{\boldsymbol{k}} - f_{\boldsymbol{k}+\boldsymbol{q}}}{\varepsilon_{\boldsymbol{k}} - \varepsilon_{\boldsymbol{k}+\boldsymbol{q}}} \ .$$

The aim of this task is to compute the Lindhard response function for $T = 0$ in one and three dimensions. From this, one finds an important qualitative difference which implies that the one dimensional electron gas at zero temperature is unstable with respect to the formation of a periodically varying electron charge or electron spin density. In addition, the Thomas-Fermi approximation is obtained as the limit of small $q = |\boldsymbol{q}|$.

(a)* Show that the Fermi function

$$f_{\boldsymbol{k}} = f(\varepsilon_{\boldsymbol{k}}, T) = \frac{1}{1 + \exp\left[\frac{\varepsilon_{\boldsymbol{k}} - \mu}{k_B T}\right]}$$

with chemical potential μ reduces to the a Heaviside step function $\theta(\varepsilon_F - \varepsilon_{\boldsymbol{k}})$ in the limit $T \to 0^+$. Note that $\mu = \varepsilon_F$ for $T = 0$ and $\varepsilon_F \equiv \varepsilon_{k_F}$.

(b)* Start with the case $d = 3$ and the dispersion of the free electron gas

$$\varepsilon_{\boldsymbol{k}} = \frac{\hbar^2 \boldsymbol{k}^2}{2m}$$

and show by explicit calculation for $T = 0$ that the response function depends only on $q = |\boldsymbol{q}|$ and is given by

$$\Pi_0^{(3d)}(q) = -n^{(3d)}(\varepsilon_F) \left[1 + \frac{1 - x^2}{2x} \ln\left|\frac{1+x}{1-x}\right|\right] \ ,$$

where $x = \frac{q}{2k_F}$ and $n^{(3d)}(\varepsilon_F) = \frac{m k_F}{(2\pi)^2 \hbar^2}$ is the density of states at the Fermi level per spin.

(c) For the case $d = 1$ assume that

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m}$$

and show by explicit calculation for $T = 0$ that the response function near $q = 2k_F$ is given by

$$\Pi_0^{(1d)}(q) = \frac{-m}{\hbar^2 q\pi} \ln\left|\frac{q+2k_F}{q-2k_F}\right| \approx -\frac{1}{2} n^{(1d)}(\varepsilon_F) \ln\left|\frac{q+2k_F}{q-2k_F}\right| \ ,$$

where $n^{(1d)}(\varepsilon_F) = \frac{m}{\pi\hbar^2 k_F}$ is the density of states at the Fermi level.

(d) Discuss the qualitative difference between the one dimensional and three dimensional result for Π_0. Especially consider the limit $q \to 2k_F$. What can you say about the derivative of $\Pi_0^{(3d)}(q)$ at $q = 2k_F$? Sketch $\Pi_0(q)/\Pi_0(0)$ as a function of q for both cases in one diagram.

In one dimension, the phenomenon observed for $q = 2k_F$ is due to so called "perfect nesting", where pairs of one filled and one empty state with energy close to ε_F reside on different branches of the dispersion. Their momentum difference is $q = 2k_F$, which equals the "nesting vector" in one dimension. The phase space volume for the creation of these excitation is much larger as compared to higher dimensional cases. This explains that the $q = 2k_F$ part contributes significantly in the momentum integral and leads to a divergence in the response function.

(e) By taking the limit $q \to 0$ in the result for $\Pi_0(q)$ show that the response function is finite and that it is essentially given by the density of state at the Fermi level

$$\Pi_0(q = 0) = \text{const.} \cdot n(\varepsilon_F) \ .$$

This is the so called Thomas-Fermi approximation.