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Advanced Statistical Physics - Problem Set 1

Summer Term 2025

Due Date: This problem set will be discussed in the seminars on Monday, April 14, and Wednesday, April 16. This week no solutions need to be handed in. The points for the set are bonus points.

1. Quantization of the radiation field

2+3+3 Bonus Points

In the absence of charges, and in the Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$, the electromagnetic field is described by the Lagrangian

$$L(t) = \frac{1}{2} \int_\Omega d^3x \left[\epsilon_0 \left(\partial_t \boldsymbol{A}\right)^2 + \frac{1}{\mu_0} \boldsymbol{A} \cdot \nabla^2 \boldsymbol{A} \right].$$

Here ϵ_0 denotes the vacuum dielectric constant, μ_0 is the vacuum permeability, and Ω is a cuboid with extensions L_x, L_y, and L_z. Note that the speed of light is $c = 1/\sqrt{\epsilon_0 \mu_0}$.

(a) Write down the Lagrange equation for $\boldsymbol{A}$.

(b) Find eigenfunctions $\boldsymbol{A}_{\boldsymbol{k}}$ and eigenvalues $\omega_{\boldsymbol{k}}^2$ of the equation

$$-\nabla^2 \boldsymbol{A}(\boldsymbol{x}) = \frac{\omega_{\boldsymbol{k}}^2}{c^2} \boldsymbol{A}(\boldsymbol{x}),$$

by using periodic boundary conditions. It may be useful to introduce, for each $\boldsymbol{k}$, a set of orthonormal vectors $\{\hat{\boldsymbol{\xi}}_{\boldsymbol{k},1}, \hat{\boldsymbol{\xi}}_{\boldsymbol{k},2}\}$ which are both perpendicular to $\boldsymbol{k}$. The time-dependent solution then has a series expansion

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{1}{\sqrt{\Omega}} \sum_{\boldsymbol{k},j} \alpha_{\boldsymbol{k},j}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\boldsymbol{\xi}}_{\boldsymbol{k},j}.$$

Insert this series expansion in the Lagrangian, and find the momenta

$$\pi_{\boldsymbol{k},i} = \frac{\partial L}{\partial \dot{\alpha}_{\boldsymbol{k},i}},$$

canonically conjugate to the coordinates $\alpha_{\boldsymbol{k},i}$. Use the Legendre transform $H = \sum_{\boldsymbol{k},i} \pi_{\boldsymbol{k},i} \dot{\alpha}_{\boldsymbol{k},i} - L(\pi_{\boldsymbol{k},i}, \alpha_{\boldsymbol{k},i})$ to obtain the Hamiltonian.

Hint: The first equation can be obtained from the Euler-Lagrange equation in (a) by using an ansatz $\mathrm{e}^{-i\omega_{\boldsymbol{k}} t} \boldsymbol{A}(\boldsymbol{x})$. *Here, assume that* $\boldsymbol{A}(\boldsymbol{x}, t)$ *is real. Using this it can be shown that* $\alpha_{-\boldsymbol{k},j} = \alpha_{\boldsymbol{k},j}^\dagger$.

(c) The classical Hamiltonian $H(\{\pi_{\boldsymbol{k},i}, \alpha_{\boldsymbol{k},i}\})$ can be quantised by imposing canonical commutation relations

$$[\alpha_{\boldsymbol{k},i}, \alpha_{\boldsymbol{q},j}] = 0, \quad [\pi_{\boldsymbol{k},i}, \pi_{\boldsymbol{q},j}] = 0, \quad [\alpha_{\boldsymbol{k},i}, \pi_{\boldsymbol{q},j}] = i\hbar\delta_{\boldsymbol{k},\boldsymbol{q}}\delta_{i,j},$$

on the coordinates $\alpha_{\boldsymbol{k},i}$ and their canonically conjugate momenta $\pi_{\boldsymbol{k},j}$. In analogy to the one-dimensional harmonic oscillator, we define photon creation and annihilation operators

$$a^{\dagger}_{\boldsymbol{k},j} = \sqrt{\frac{\epsilon_0\omega_{\boldsymbol{k}}}{2\hbar}}\left(\alpha_{-\boldsymbol{k},j} - \frac{i}{\epsilon_0\omega_{\boldsymbol{k}}}\pi_{\boldsymbol{k},j}\right), \quad a_{\boldsymbol{k},j} = \sqrt{\frac{\epsilon_0\omega_{\boldsymbol{k}}}{2\hbar}}\left(\alpha_{\boldsymbol{k},j} + \frac{i}{\epsilon_0\omega_{\boldsymbol{k}}}\pi_{-\boldsymbol{k},j}\right).$$

Show that $a_{\boldsymbol{k},j}$ and $a^{\dagger}_{\boldsymbol{k},j}$ obey the commutation relations of harmonic oscillator ladder operators, and express the Hamiltonian in terms of $a_{\boldsymbol{k},j}$ and $a^{\dagger}_{\boldsymbol{k},j}$.