Institut for Theoretical Physics
University Leipzig

Prof. Dr. B. Rosenow
M. Kühn

Statistical Mechanics of Deep Learning - Problem set 12

Winter Term 2024/25

The problem set will be discussed in the seminar on **Monday 20.01.2024, 9:15**.

22. The Langevin equation

3 + 3 Points

The Langevin equation in $1d$ is given by

$$m\ddot{x} = -\frac{1}{\mu}\dot{x} + f(t),$$

where the force is uncorrelated in time

$$\langle f(t)f(t')\rangle = 2DT\delta(t-t').$$

(a) Show that

$$x(\omega) = \frac{-\mu}{i\omega + m\mu\omega^2} f(w),$$

where the Fourier transform of a function $g(t)$ is defined as

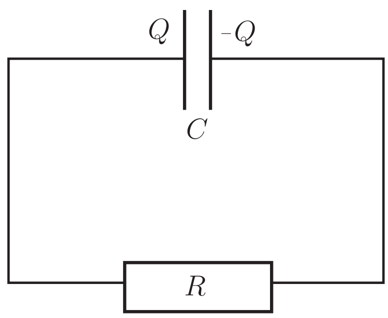
$$g(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} g(t).$$

(b) Calculate the velocity correlation function $\langle \dot{x}(t)\dot{x}(t')\rangle$.

Hint: use contour integration and the residue theorem to evaluate the Fourier transform of the function $1/(1+\omega^2)$.

23. Johnson–Nyquist noise

2+2+2+1 Points



Consider a circuit with a capacitor C and resistor R in series as depicted in the figure. Demanding that the voltage Q/C on the capacitor and IR (with $I = \dot{Q}$) on the resistor are equal to each other, we find the equation of motion for the charge Q

$$\frac{Q(t)}{C} = -\dot{Q}(t)R + \delta U(t) \ ,$$

where $\delta U(t)$ describes fluctuations in the voltage due to thermal noise in the resistor. Here, $\delta U(t)$ has zero average and is uncorrelated in time

$$\langle \delta U(t)\rangle = 0 \text{ and } \langle \delta U(t)\delta U(t')\rangle = \lambda\delta(t-t') \ .$$

(a) Show that the solution for the time dependence of the charge Q is given by

$$Q(t) = Q(t_0)\mathrm{e}^{-\frac{t-t_0}{RC}} + \frac{1}{R}\int_{t_0}^{t} d\tau \ \mathrm{e}^{\frac{1}{RC}(\tau - t)} \ \delta U(\tau) \ .$$

Hint: You may start comparing $\frac{d}{dt}[\mathrm{e}^{+t/\tau_0}Q(t)]$ to the differential equation for $Q(t)$.

(b) Use the result from (a) to compute $\langle Q(t)\rangle$ and show that

$$\langle [Q(t) - \langle Q(t)\rangle]^2\rangle = \frac{\lambda C}{2R}\left[1 - \exp\left(\frac{2}{RC}(t_0 - t)\right)\right] .$$

(c) Calculate $\langle Q^2\rangle$ using standard statistical mechanics with the Hamiltonian for the electrical circuit

$$\mathcal{H} = \frac{Q^2}{2C} .$$

Compare this result with the infinite time limit $\lim_{t\to\infty} \langle [Q(t) - \langle Q(t)\rangle]^2\rangle$ from (b) to determine the noise strength λ.

(d) Compute the zero frequency noise

$$\int_{-\infty}^{\infty} dt \, \langle \delta U(t)\delta U(0)\rangle .$$