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Advanced Quantum Mechanics - Problem Set 7

Winter Term 2022/23

Due Date: Hand in solutions to problems marked with * to mailbox 39 with label "Advanced Quantum Mechanics" inside ITP room 105b before the lecture on **Friday, 02.12.2022, 09:15**. The problem set will be discussed in the tutorials on Monday 05.12.2022 and Wednesday 07.12.2022.

1. Graphene

3+1+3 Points

The Hamiltonian for graphene near the $\boldsymbol{K}'$ point is given by

$$H = \hbar v_F \begin{pmatrix} 0 & q_x + iq_y \\ q_x - iq_y & 0 \end{pmatrix},$$

where v_F is the Fermi velocity.

(a) Calculate the normalized eigenstates of this Hamiltonian.

(b) We now consider next-nearest neighbors. The Hamiltonian is then modified by

$$H_{\text{nnn}} = -\frac{t'}{2} \sum_{\langle\langle i,j \rangle\rangle} (|i, A\rangle\langle j, A| + |i, B\rangle\langle j, B| + \text{h.c}),$$

where A and B denote different sub-lattices and the sum is over next-nearest neighbors. Write down the next-nearest neighbor lattice vectors.

(c) Show that the next-nearest neighbors give rise to an extra contribution to the spectrum of $-t' f(\boldsymbol{q})$ with

$$f(\boldsymbol{q}) = 2\cos\left(\sqrt{3} q_y a\right) + 4\cos\left(\frac{\sqrt{3}}{2} q_y a\right)\cos\left(\frac{3}{2} q_x a\right).$$

2. Free particle solutions of the Dirac equation

3 Points

Calculate the eigenvalues of the free-particle Dirac equation

$$\begin{pmatrix} m & 0 & p & 0 \\ 0 & m & 0 & -p \\ p & 0 & -m & 0 \\ 0 & -p & 0 & -m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = E \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

*3. Commutators of Dirac matrices

2+2 Points

Consider the Dirac matrices

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},$$

where $\boldsymbol{\sigma}$ is the vector of Pauli matrices and $\mathbb{1}_2$ is the 2-dimensional unit matrix. Define also

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

Show that (i) $\beta\Sigma_i = \Sigma_i\beta$, and that (ii) $[\alpha_i, \Sigma_j] = 2i\epsilon_{ijk}\alpha_k$.

*4. Four-current for the free particle solutions of the Dirac equation

1+4 Points

The free particle solutions of the Dirac equation can be written using

$$\boldsymbol{u}_R^{(+)}(p) = \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E_p+m} \\ 0 \end{pmatrix}, \qquad \boldsymbol{u}_L^{(+)}(p) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{E_p+m} \end{pmatrix},$$

for solutions with positive energy $E = E_p$, and

$$\boldsymbol{u}_R^{(-)}(p) = \begin{pmatrix} \frac{-p}{E_p+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{u}_L^{(-)}(p) = \begin{pmatrix} 0 \\ \frac{p}{E_p+m} \\ 0 \\ 1 \end{pmatrix},$$

for solutions with negative energy $E = -E_p$.

(a) What are the free-particle wave-functions?

(b) Calculate the four-current $j^\mu = \bar{\Psi}\gamma^\mu\Psi$, where $\bar{\Psi} = \Psi^\dagger\beta$. Interpret your result.