Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow
Dr. S. Fischer

Advanced Quantum Mechanics - Problem Set 14

Winter Term 2021/22

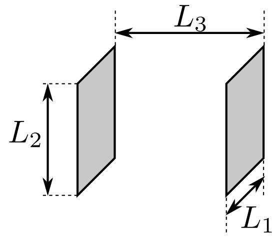
Due Date: Problems on this exercise sheet are not mandatory. Instead, points scored here are counted on top of points already reached for completing mandatory exercises. Hand in solutions to problems marked with * before **Monday, 14.02.2022, 12:00**.

*1. Casimir Effect

4+4+2+1 Points

As shown in Problem Set 10 Exercise 2, the Hamiltonian of the quantized radiation field confined to a box with volume $V = L_1 L_2 L_3$ and with periodic boundary conditions, is given by

$$H = \sum_{\boldsymbol{k}} \sum_{\lambda=\pm} \hbar\omega_{\boldsymbol{k}} \left(a^\dagger_{\boldsymbol{k},\lambda} a_{\boldsymbol{k},\lambda} + \frac{1}{2} \right) \,, \quad \omega_{\boldsymbol{k}} = c|\boldsymbol{k}| \,, \quad k_i = \frac{\pi}{L_i} n_i \,, \quad n_i \in \mathbb{N} \,.$$



In particular we found that the ground state, in which no modes are excited, has a divergent energy. Whilst this divergent vacuum zero-point energy is not observable, the dependence on the boundaries does lead to observable phenomena.
To investigate this, we consider in the following two conducting plates with surface areas $A = L_1 L_2$ separated by a distance L_3. In the plane of the plates we will still be using periodic boundary conditions and consider the limit $L_1, L_2 \to \infty$. Since the electric field $\boldsymbol{E}$ on the plates vanishes, only modes with $|\boldsymbol{E}| \propto \sin(k_3 x_3)$ are possible. Here $k_3 = n_3\pi/L_3$ with $n_3 = 1, 2, \ldots$. To get a finite vacuum energy we will moreover introduce an exponential cutoff $e^{-\epsilon\omega_{\boldsymbol{k}}}$ with $\epsilon > 0$, and take the limit of $\epsilon \to 0$ at the end of the calculation. The energy density per unit plate area between the plates is given by

$$\begin{aligned}\sigma_E(L_3) &= \lim_{L_1, L_2 \to \infty} \frac{1}{L_1 L_2} \sum_{\boldsymbol{k}} \hbar\omega_{\boldsymbol{k}}\, e^{-\epsilon\omega_{\boldsymbol{k}}} \\ &= \hbar c \sum_{n_3=1}^{\infty} \int \frac{d^2k}{(2\pi)^2} \sqrt{k_1^2 + k_2^2 + (\tfrac{\pi n_3}{L_3})^2}\, e^{-\epsilon c \sqrt{k_1^2 + k_2^2 + (\frac{\pi n_3}{L_3})^2}}\end{aligned}$$

(a) Using polar coordinates and a suitable subsitution show that $\sigma_E(L_3)$ can be written as

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2} \frac{\partial^2}{\partial\epsilon^2} \sum_{n=1}^{\infty} \int_{n\pi c/L_3}^{\infty} d\omega e^{-\epsilon\omega} \,.$$

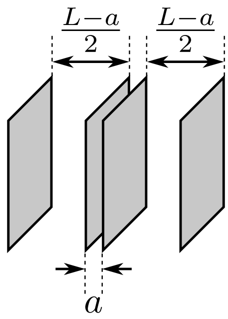
(b) Calculate the integral over ω and perform the sum to show that

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2} \frac{\partial^2}{\partial\epsilon^2} \Big(\frac{1}{\epsilon} \frac{1}{e^{\epsilon\pi c/L_3} - 1} \Big) \,.$$

Show further that

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2}\Big(\frac{6}{\epsilon^4}\frac{L_3}{\pi c} - \frac{1}{\epsilon^3} - \frac{1}{360}\Big(\frac{\pi c}{L_3}\Big)^3 + \mathcal{O}(\epsilon^2)\Big).$$

(c) The energy density calculated in the previous part diverges as the distance between the plates increases ($L_3 \to \infty$). This will be our reference point. We therefore consider two plates separated by a fixed distance a, together with two external plates which are places a further distance $(L-a)/2$ away. The relevant energy density is then given by



$$\sigma_E(a, L) = \sigma_E(a) + 2\sigma_E\left(\frac{L-a}{2}\right).$$

Find an expression for $\sigma_E(a, L)$ using your result in (b).

(d) Since the energy density varies with the distance between plates, the plates experience a pressure which is given by

$$p_{\text{vac}} = -\lim_{L\to\infty}\frac{\partial}{\partial a}\sigma_E(a, L).$$

How large is this pressure for $A = 1\,\text{cm}^2$ and $a = 1\,\mu\text{m}$?

*2. Zitterbewegung

2+2+2 Points

In this problem we will consider the Dirac Hamiltonian

$$\hat{H}_D = c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2,$$

where m is the mass of the particle, c is the speed of light, and $\boldsymbol{\alpha}$ and β are matrices given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

with $\boldsymbol{\sigma}$ denoting the vector of Pauli matrices and I_2 denoting the 2×2 unit matrix. The Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that the velocity operator is given by $\hat{\boldsymbol{v}} = c\boldsymbol{\alpha}$.

Hint: You may use the Heisenberg equation of motion which states that an operator $\hat{A}$ which does not explicitly depend on time satisfies $-i\hbar\dot{\hat{A}} = [\hat{H}, \hat{A}]$.

(b) Consider now a Dirac particle at rest in a volume V. A general eigenspinor can then be written as

$$\psi = \frac{1}{\sqrt{2V}} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imc^2 t/\hbar} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{imc^2 t/\hbar} \right].$$

Give a physical interpretation of the two terms in the spinor.

(c) Derive an expression for $\langle \hat{v}_z \rangle = \langle \psi | \hat{v}_z | \psi \rangle$ using the spinor defined in the previous part of the problem. Comment on your result.