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Advanced Quantum Mechanics - Problem Set 10

Winter Term 2021/22

Due Date: Hand in solutions to problems marked with * before **Monday, 10.01.2022, 12:00**. The problem set will be discussed in the tutorials on Wednesday, 12.01.2022, and Friday, 14.01.2022

1. Addition of three angular momenta

4+4+2 Points

j	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$j_1 + 1$	$\left[\frac{(j_1+m)(j_1+m+1)}{(2j_1+1)(2j_1+2)}\right]^{1/2}$	$\left[\frac{(j_1-m+1)(j_1+m+1)}{(2j_1+1)(j_1+1)}\right]^{1/2}$	$\left[\frac{(j_1-m)(j_1-m+1)}{(2j_1+1)(2j_1+2)}\right]^{1/2}$
j_1	$-\left[\frac{(j_1+m)(j_1-m+1)}{2j_1(j_1+1)}\right]^{1/2}$	$\left[\frac{m^2}{j_1(j_1+1)}\right]^{1/2}$	$\left[\frac{(j_1-m)(j_1+m+1)}{2j_1(j_1+1)}\right]^{1/2}$
$j_1 - 1$	$\left[\frac{(j_1-m)(j_1-m+1)}{2j_1(2j_1+1)}\right]^{1/2}$	$-\left[\frac{(j_1-m)(j_1+m)}{j_1(2j_1+1)}\right]^{1/2}$	$\left[\frac{(j_1+m+1)(j_1+m)}{2j_1(2j_1+1)}\right]^{1/2}$

Figure 1: Table of Clebsch-Gordan coefficients (from B. H. Bransden and C. J. Joachain). The table should be understood as a matrix as discussed in the lecture, with the convention $\sqrt{x^2} = x$.

Consider three angular momenta $\hat{\boldsymbol{L}}_1$, $\hat{\boldsymbol{L}}_2$, and $\hat{\boldsymbol{L}}_3$ with $l_1 = l_2 = l_3 = 1$.

(a) First, add the two angular momenta $\hat{\boldsymbol{L}}_1$ and $\hat{\boldsymbol{L}}_2$, with $l_1 = l_2 = 1$ and m_1, m_2, to a total angular momentum $\hat{\boldsymbol{L}}$, with l and m. Use the above table to show that

$$
\begin{aligned}
|l=1, m=\ \ 1\rangle &= \frac{1}{\sqrt{2}}\Big(+|m_1=1; m_2=0\rangle - |m_1=0; m_2=1\rangle\Big) \\
|l=1, m=-1\rangle &= \frac{1}{\sqrt{2}}\Big(+|m_1=0; m_2=-1\rangle - |m_1=-1; m_2=0\rangle\Big) \\
|l=1, m=\ \ 0\rangle &= \frac{1}{\sqrt{2}}\Big(+|m_1=1; m_2=-1\rangle - |m_1=-1; m_2=1\rangle\Big) \\
|l=0, m=\ \ 0\rangle &= \frac{1}{\sqrt{3}}\Big(+|m_1=1; m_2=-1\rangle - |m_1=0; m_2=0\rangle + |m_1=-1; m_2=1\rangle\Big),
\end{aligned}
$$

where $|m_1; m_2\rangle \equiv |l_1 = 1, m_1; l_2 = 1, m_2\rangle$. Compare these results to the results you obtained for Problem 1 of Problem Set 9.

(b) Add all three angular momenta to get a state with total angular momentum $l = 0$.

Hint: First add $\hat{\boldsymbol{L}}_1$ and $\hat{\boldsymbol{L}}_2$, and then add $\hat{\boldsymbol{L}}_3$ to the resulting angular momentum. Use the same basis for adding $\hat{\boldsymbol{L}}_1$ and $\hat{\boldsymbol{L}}_2$ as in Problem 1 of Problem Set 9, but don't keep all 27 basis functions. Instead keep only the functions that together with $\hat{\boldsymbol{L}}_3$ can add to $l = 0$. The result from (a) might be helpful.

(c) Show that this state can be written as a 3×3 determinant and that it therefore is anti-symmetric.

Hint: You can write $|m_1; m_2; m_3\rangle = |m_1\rangle \otimes |m_2\rangle \otimes |m_3\rangle = |m_1\rangle|m_2\rangle|m_3\rangle$.

*2. Quantisation of the Radiation Field *2+3+3 Points*

In the absence of charges, and in the Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$, the electromagnetic field is described by the Lagrangian

$$L(t) = \frac{1}{2} \int_\Omega d^3x \left[\epsilon_0 \left(\partial_t \boldsymbol{A}\right)^2 + \frac{1}{\mu_0} \boldsymbol{A} \cdot \nabla^2 \boldsymbol{A} \right].$$

Here ϵ_0 denotes the vacuum dielectric constant, μ_0 is the vacuum permeability, and Ω is a cuboid with extensions L_x, L_y, and L_z. Note that the speed of light is $c = 1/\sqrt{\epsilon_0 \mu_0}$.

(a) Write down the Lagrange equation for $\boldsymbol{A}$.

(b) Find eigenfunctions $\boldsymbol{A}_{\boldsymbol{k}}$ and eigenvalues $\omega_{\boldsymbol{k}}^2$ of the equation

$$-\nabla^2 \boldsymbol{A}(\boldsymbol{x}) = \frac{\omega_{\boldsymbol{k}}^2}{c^2} \boldsymbol{A}(\boldsymbol{x}),$$

by using periodic boundary conditions. It may be useful to introduce, for each $\boldsymbol{k}$, a set of orthonormal vectors $\{\hat{\boldsymbol{\xi}}_{\boldsymbol{k},1}, \hat{\boldsymbol{\xi}}_{\boldsymbol{k},2}\}$ which are both perpendicular to $\boldsymbol{k}$. The time-dependent solution then has a series expansion

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{1}{\sqrt{\Omega}} \sum_{\boldsymbol{k},j} \alpha_{\boldsymbol{k},j}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{\boldsymbol{\xi}}_{\boldsymbol{k},j}.$$

Insert this series expansion into the Lagrangian, and find the momenta

$$\pi_{\boldsymbol{k},i} = \frac{\partial L}{\partial \dot{\alpha}_{\boldsymbol{k},i}},$$

canonically conjugate to the coordinates $\alpha_{\boldsymbol{k},i}$. Use the Legendre transform $H = \sum_{\boldsymbol{k},i} \pi_{\boldsymbol{k},i} \dot{\alpha}_{\boldsymbol{k},i} - L(\pi_{\boldsymbol{k},i}, \alpha_{\boldsymbol{k},i})$ to obtain the Hamiltonian.

Hint: The first equation can be obtained from the Euler-Lagrange equation in (a) by using that $\boldsymbol{A}(\boldsymbol{x}, t) = \mathrm{e}^{-i\omega_{\boldsymbol{k}} t} \boldsymbol{A}(\boldsymbol{x})$. *Here, assume that* $\boldsymbol{A}(\boldsymbol{x})$ *is real. Using this it can be shown that* $\alpha_{-\boldsymbol{k},j} = \alpha_{\boldsymbol{k},j}^\dagger$.

(c) The classical Hamiltonian $H(\{\pi_{\boldsymbol{k},i}, \alpha_{\boldsymbol{k},i}\})$ can be quantised by imposing canonical commutation relations

$$[\alpha_{\boldsymbol{k},i}, \alpha_{\boldsymbol{q},j}] = 0, \quad [\pi_{\boldsymbol{k},i}, \pi_{\boldsymbol{q},j}] = 0, \quad [\alpha_{\boldsymbol{k},i}, \pi_{\boldsymbol{q},j}] = i\hbar \delta_{\boldsymbol{k},\boldsymbol{q}} \delta_{i,j},$$

on the coordinates $\alpha_{\boldsymbol{k},i}$ and their canonically conjugate momenta $\pi_{\boldsymbol{k},j}$. In analogy to the one-dimensional harmonic oscillator, we define photon creation and annihilation operators

$$a_{\boldsymbol{k},j}^\dagger = \sqrt{\frac{\epsilon_0 \omega_{\boldsymbol{k}}}{2\hbar}} \left(\alpha_{-\boldsymbol{k},j} - \frac{i}{\epsilon_0 \omega_{\boldsymbol{k}}} \pi_{\boldsymbol{k},j} \right), \quad a_{\boldsymbol{k},j} = \sqrt{\frac{\epsilon_0 \omega_{\boldsymbol{k}}}{2\hbar}} \left(\alpha_{\boldsymbol{k},j} + \frac{i}{\epsilon_0 \omega_{\boldsymbol{k}}} \pi_{-\boldsymbol{k},j} \right).$$

Show that $a_{\boldsymbol{k},j}$ and $a_{\boldsymbol{k},j}^\dagger$ obey the commutation relations of harmonic oscillator ladder operators, and express the Hamiltonian in terms of $a_{\boldsymbol{k},j}$ and $a_{\boldsymbol{k},j}^\dagger$.