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Advanced Quantum Mechanics - Problem Set 6

Winter Term 2021/22

Due Date: Hand in solutions to problems marked with * before **Monday, 29.11.2021, 12:00**. The problem set will be discussed in the tutorials on Wednesday, 01.12.2021, and Friday, 03.12.2021

*1. Graphene

3+1+3 Points

The Hamiltonian for graphene near the $\boldsymbol{K}'$ point is given by

$$H = \hbar v_F \begin{pmatrix} 0 & q_x + iq_y \\ q_x - iq_y & 0 \end{pmatrix},$$

where v_F is the Fermi velocity.

(a) Calculate the normalized eigenstates of this Hamiltonian.

(b) We now consider next-nearest neighbors. The Hamiltonian is then modified by

$$H_{\text{nnn}} = -\frac{t'}{2} \sum_{\langle\langle i,j \rangle\rangle} (|i,A\rangle\langle j,A| + |i,B\rangle\langle j,B| + \text{h.c}),$$

where A and B denote different sub-lattices and the sum is over next-nearest neighbors. Write down the next-nearest neighbor lattice vectors.

(c) Show that the next-nearest neighbors give rise to an extra contribution to the spectrum of $-t'f(\boldsymbol{q})$ with

$$f(\boldsymbol{q}) = 2\cos\left(\sqrt{3}q_y a\right) + 4\cos\left(\frac{\sqrt{3}}{2}q_y a\right)\cos\left(\frac{3}{2}q_x a\right).$$

2. Relativistic Landau Levels

3+3+2 Points

A Hamiltonian for electrons moving in two spatial dimensions is given by

$$H = v_F \begin{pmatrix} -\boldsymbol{\sigma}^* \cdot \boldsymbol{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \boldsymbol{p} \end{pmatrix},$$

where v_F is the Fermi velocity, $\boldsymbol{p}$ the momentum and $\boldsymbol{\sigma}$ the vector of Pauli matrices. The eigenstates can be written as four-dimensional state vectors with contributions from the $\boldsymbol{K}$ and $\boldsymbol{K}'$ points. That is, we write

$$\boldsymbol{\chi} = \begin{pmatrix} \chi'_A \\ \chi'_B \\ \chi_A \\ \chi_B \end{pmatrix}.$$

(a) Show that the eigenvalue equations decouple into

$$E^2\chi_A = v_F^2(p_x - ip_y)(p_x + ip_y)\chi_A,$$
$$E^2\chi_B = v_F^2(p_x + ip_y)(p_x - ip_y)\chi_B,$$

and similar for the primed parts of the eigenstates.

(b) Suppose now a magnetic field is switched on. Using the Landau gauge $\boldsymbol{A} = (-By, 0)$, perform the minimal substitution $\boldsymbol{p} \to \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}$ in the eigenvalue equations in part (a) and deduce the form of the eigenfunctions.

(c) What does the energy spectrum look like?

3. Representations of γ matrices *2+1+2 Points*

(Note that for bachelor students this problem is not mandatory in the course with the lower number of credit points.)

The γ matrices can be written as

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3$$
$$\gamma_0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},$$

where σ_i denotes a Pauli matrix and $\mathbb{1}_n$ the $n \times n$ unit matrix. Consider the metric $\eta = \text{diag}(1, -1, -1, -1)$.

(a) Show that the γ matrices satisfy the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{1}_4$, $\mu, \nu \in \{0, 1, 2, 3\}$.

(b) A different representation is the Weyl representation where

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

Show that these still satisfy the Clifford algebra.

(c) Using only the Clifford algebra and properties of the trace show that $\text{tr}(\gamma_\mu) = 0$, $\text{tr}(\gamma_\mu\gamma_\nu) = 4\eta_{\mu\nu}$, and $\text{tr}(\gamma_\mu\gamma_\nu\gamma_\rho) = 0$.