
Advanced Quantum Mechanics - Problem Set 0

Winter Term 2021/22

Due Date: Hand in solutions to problems marked with * before **Monday, 18.10.2021, 12:00 (noon)**, via Moodle. The problem set will be discussed in the tutorials on Wednesday, 20.10.2021, and Friday, 22.10.2021.

Internet: [Advanced Quantum Mechanics exercises](#)

The aim of the problem set is to familiarize yourself with Dirac notation.

*1. Two-level system

3 Points

Consider the Hamiltonian of a two-level system

$$\hat{H} = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|),$$

where $a > 0$ has the dimension of an energy. Calculate the energy eigenvalues and eigenstates with respect to the orthonormal basis $\{|1\rangle, |2\rangle\}$.

2. Unitary transformation

1+2 Points

Consider the unitary transformation $|\psi'\rangle = \hat{U}|\psi\rangle$.

- (a) Show that the operator \hat{A} has to be transformed as $\hat{A}' = \hat{U}\hat{A}\hat{U}^\dagger$
- (b) Show that with the above definitions the following properties of the operators are conserved in the transformation:
 - (i) linearity and hermiticity
 - (ii) commutation relations
 - (iii) the eigenvalue spectrum
 - (iv) the algebraic relations $\hat{F} = \hat{K} + \hat{M}$ and $\hat{F} = \hat{K}\hat{M}$

3. Momentum representation

2+2 Points

Let $|\alpha\rangle$ and $|\beta\rangle$ be arbitrary ket-vectors. Use the normalization $\langle p|p'\rangle = \delta(p-p')$ and completeness relation $\int dx |x\rangle\langle x| = \hat{1}$ to obtain an expression for $\langle x|p\rangle$. Show then explicitly

(a) $\langle p|\hat{x}|\alpha\rangle = i\hbar \frac{\partial}{\partial p} \psi_\alpha(p),$

(b) $\langle \beta|\hat{x}|\alpha\rangle = \int dp \psi_\beta^*(p) i\hbar \frac{\partial}{\partial p} \psi_\alpha(p).$

Here $\psi_\alpha(p) = \langle p|\alpha\rangle$ and $\psi_\beta(p) = \langle p|\beta\rangle$ are one-dimensional wave functions in momentum representation and \hat{x} is the position operator.

Hint: Use the Fourier-representation of the delta function: $\delta(x-x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \exp [i(x-x')y].$

4. Change of representation

4 Points

Let us denote the eigenstate of the position operator \hat{x} with eigenvalue x as $|x\rangle$, the eigenstate of the momentum operator \hat{p} with eigenvalue p as $|p\rangle$ and the eigenstate of the Hamilton operator $\hat{H} = \frac{\hat{p}^2}{2m}$ with energy E as $|E\rangle$. Consider a particle in the state $|\Psi\rangle$ which in the momentum representation is given by $\langle p|\Psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp(-ix_0 \frac{p}{\hbar})$.

(a) Calculate $\langle x|\Psi\rangle$. How can the state $|\Psi\rangle$ therefore be physically interpreted?

(b) Use the eigenvalue equation for \hat{H} and the matrix elements $\langle x|\hat{H}|x'\rangle = -\frac{\hbar^2}{2m} \delta(x-x')$ to derive a differential equation for $\Psi_E(x) = \langle x|E\rangle$ and obtain $\Psi_E(x)$.

(c) Use your results obtained in (b) to calculate $\langle p|E\rangle$ and express the eigenvalues E in terms of the eigenvalues p .