Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow
M. Thamm

Quantum Field Theory of Many-Particle Systems - Problem Set 9

Summer Semester 2024

Due: The problem set will be discussed in the tutorial on **Friday, 07.06.2024, 13:30**.

Internet: The problem sets can be downloaded from
https://home.uni-leipzig.de/stp/QFT_of_MPS_SS24.html

1. Polarization propagator in one dimension

3+3+6 Punkte

The polarization propagator in one spatial dimension is defined as

$$\Pi(i\omega_n, q) = 2\int \frac{dk}{2\pi} T \sum_{\epsilon_l} \frac{1}{i\epsilon_l + i\omega_n - \xi(k+q)} \frac{1}{i\epsilon_l - \xi(k)},$$

where $\xi(k) = \hbar^2 k^2/(2m) - \mu$ is the kinetic energy with respect to the chemical potential, ω_n is a bosonic Matsubara frequency, and the sum runs over fermionic Matsubara frequencies ϵ_l.

(a) As a first step towards evaluating the frequency sum, decompose the product of Green functions in the definition of $\Pi(i\omega_n, q)$ into partial fractions.

(b) Perform the frequency sum using the identity

$$T \sum_{\epsilon_l} \frac{e^{i\epsilon_l \eta}}{i\epsilon_l - \xi} = n_F(\xi),$$

where $n_F(\xi) = 1/(e^{\beta\xi} + 1)$ denotes the Fermi distribution function.

(c) In order to perform the remaining momentum integral

$$-2\int \frac{dk}{2\pi} \frac{n_F[\xi(k+q)] - n_F[\xi(k)]}{i\omega_n - \xi(k+q) + \xi(k)},$$

split the integral in two and perform a suitable shift of the integration variable. Finally take the limit of zero temperature and use the relation $n_F(x) = \Theta(-x)$ valid in this limit.

2. Lifetime of quasi-particles *3+5 Punkte*

The lifetime of quasi-particles can be calculated from the retarded self-energy as

$$\frac{1}{\tau} \propto -2\mathrm{Im}\Sigma^R(\omega, \boldsymbol{k}_F).$$

The leading frequency dependence of the self-energy can be computed using the Fock contribution discussed in Problem Set 7. We shall thus take the zero-temperature retarded self-energy to be given by

$$\Sigma^R(\omega, \boldsymbol{k}) = 2\int_0^\omega \frac{d\epsilon}{2\pi} \int \frac{d^3q}{(2\pi)^3} \frac{V(\epsilon, \boldsymbol{q})}{\omega - \epsilon - \xi(\boldsymbol{k}-\boldsymbol{q}) + i\eta},$$

where $V(\epsilon, \boldsymbol{q})$ is an interaction between fermions and bosons which in this problem will be taken to be the screened Coulomb interaction. The upper cutoff in the frequency integral is due to the Pauli principle.

(a) In lectures it has been shown that the polarization, for small momenta and $|\epsilon| \ll v_F q$, can be written to leading order as $\Pi(\epsilon, q) = \Pi_0 + \Pi_L(\epsilon, q)$, where $\Pi_L(\epsilon, q) = \pi|\epsilon|\rho_F/(2v_F q)$ contains the leading dynamic contribution. Using that $V = V_0/(1 - \Pi V_0)$, where $V_0 = 4\pi e^2/q^2$ denotes the bare Coulomb potential, show by expanding to leading order in Π_L that the leading dynamic contribution to the interaction becomes

$$V = V_{\mathrm{scr}}^2 \Pi_L(\epsilon, q).$$

In this expression we have defined the static, screened Coulomb potential as $V_{\mathrm{scr}} = V_0/(1 - \Pi_0 V_0)$.

(b) Taking the imaginary part before evaluating the integral, and using the identity

$$\lim_{\eta \to 0^+} \mathrm{Im}\frac{1}{x + i\eta} = -\pi\delta(x),$$

calculate the leading frequency dependence of the lifetime of the quasi-particles.

Hint: Start by performing the angular integration. In order to calculate the leading frequency behaviour of the life-time it is sufficient to set all frequencies to zero in the argument of the $\delta-$function (why?). The radial integral can be performed exactly but in order to get an estimate it is also enough to assume that $q \ll \kappa$ where κ is the screening length given by $\kappa^2 = 4\pi e^2 \rho_F$.