Mathematical Methods of Modern Physics

SS 2024

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§1 Complex Functions

1.1 Complex Numbers

Def.: The set $\mathbb{C}$ of complex numbers is (as a set) equivalent to $\mathbb{R}^2$. Elements of $\mathbb{C}$ can be represented as tuples (x, y) with $x, y \in \mathbb{R}$.

In $\mathbb{C}$ we define addition and multiplication as follows:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

$(\mathbb{C}, +, \cdot)$ is a field (homework problem)

Difference between $\mathbb{R}$ and $\mathbb{C}$: there is no order relation in $\mathbb{C}$.

Consider now $(x, 0)$

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$

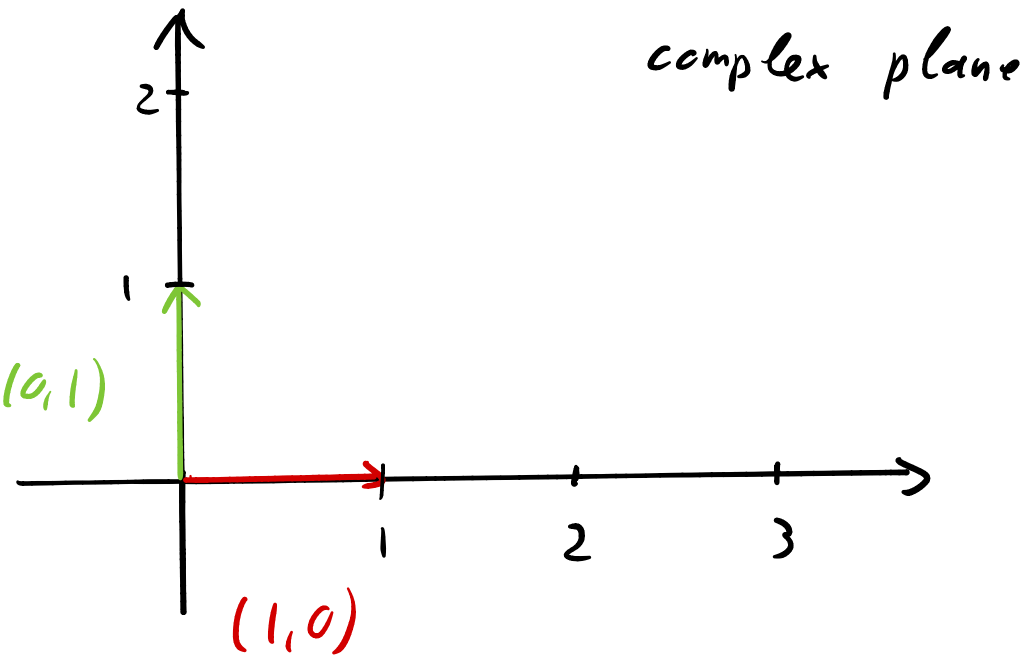
$$(x_1, 0) \cdot (x_2, 0) = (x_1 \cdot x_2, 0)$$

$\Rightarrow \quad \{(x,0) : x \in \mathbb{R}\} = \mathbb{R}$

Instead of $(x,0)$ we can simply write x.

Similarly, $(a,0)\cdot(x,y) = (ax, ay) = a(x,y)$

corresponds to scalar multiplication in the vector space $\mathbb{R}^2$.



Every complex number can be represented as

$$(x,y) = x\cdot(1,0) + y\,(0,1)$$

$$(1,0) \equiv 1 \qquad\qquad (0,1) \equiv i$$

$$(x,y) = x\cdot 1 + y\cdot i = x + iy$$

i is called imaginary unit.

$$i^2 = (0,1)\cdot(0,1) = (-1,0) = -1$$

$$(x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + x_1 i y_2 + i y_1 x_2 + i^2 y_1 y_2$$

$$= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$$

This is the motivation for the definition of multiplication given above.

Def.: Let $z = (x, y) = x + iy \in \mathbb{C}$. Then we call $x = Re(z)$ the real part of z and $y = Im(z)$ the imaginary part, and $\bar{z} = x - iy$ the conjugate complex number.

Rules: $\bar{\bar{z}} = z$

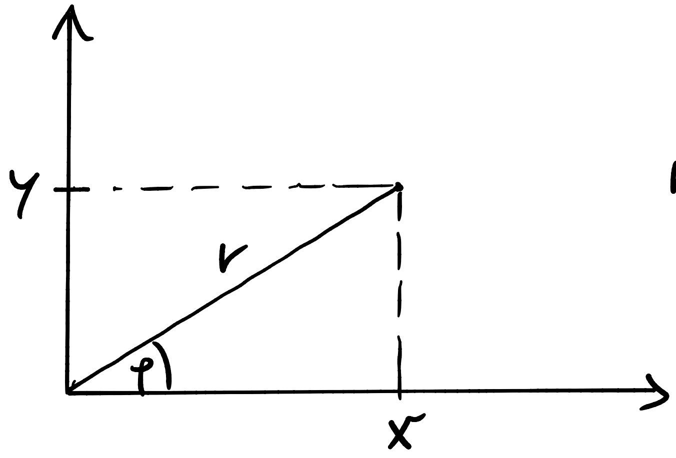
$$Re(z) = \frac{1}{2}(z + \bar{z})$$

$$Im(z) = \frac{1}{2i}(z - \bar{z})$$

$$z \cdot \bar{z} = (x + iy)(x - iy) = x^2 + y$$

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{x^2+y^2}\bar{z}$$

Polar coordinate representation of complex numbers



$$r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} =: |z|$$

$$\tan\varphi = \frac{y}{x} \quad \Rightarrow \quad \varphi = \arctan\frac{y}{x} =: \arg z$$

argument of z

$$z = x + iy = r\cos\varphi + ir\sin\varphi = r(\cos\varphi + i\sin\varphi)$$

$\varphi = \arg z$ is unique only modulo 2π

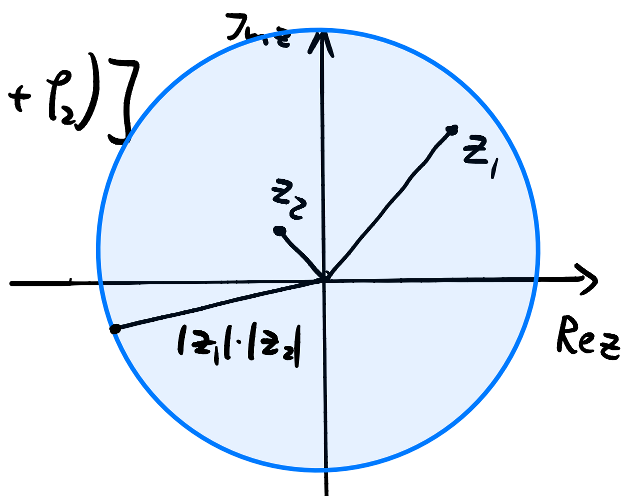
Example: polar coordinate representation of $z_1 \cdot z_2$

$$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$$

$$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$$

$$z_1 \cdot z_2 = r_1 r_2 \left[\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i(\cos\varphi_1\sin\varphi_2 + \cos\varphi_2\sin\varphi_1)\right]$$

$$= r_1 r_2 \left[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right]$$



$$z = r(\cos\varphi + i\sin\varphi)$$

$$z^2 = r^2(\cos 2\varphi + i\sin 2\varphi)$$

$$\boxed{z^n = r^n(\cos n\varphi + i\sin n\varphi)} \qquad (*)$$

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \ldots + \ldots = 1 + \frac{(iy)^2}{2!} + \frac{(iy)^4}{4!} + \ldots$$

$$i\sin y = i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \ldots + \ldots\right) = iy + \frac{(iy)^3}{3!} + \frac{(iy)^5}{5!} + \ldots$$

$$\Rightarrow \cos y + i\sin y = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = e^{iy}$$

Using this relation, we can write $z = r e^{i\varphi}$

Proof of $(*)$ $\quad z^n = (r e^{i\varphi})^n = r^n e^{in\varphi} = r^n(\cos n\varphi + i\sin n\varphi)$

$\cos\varphi + i\sin\varphi$ lies on the unit circle

Example: quotient of two complex numbers in polar coordinates

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)} = \frac{r_1}{r_2}\left[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right]$$

Example: n-th root

Let $z = r(\cos\varphi + i\sin\varphi) = r[\cos(\varphi + 2\pi k) + i\sin(\varphi + 2\pi k)]$

with $k \in \mathbb{Z}$

Find complex number $w \equiv \sqrt[n]{z}$ with $w^n = z$

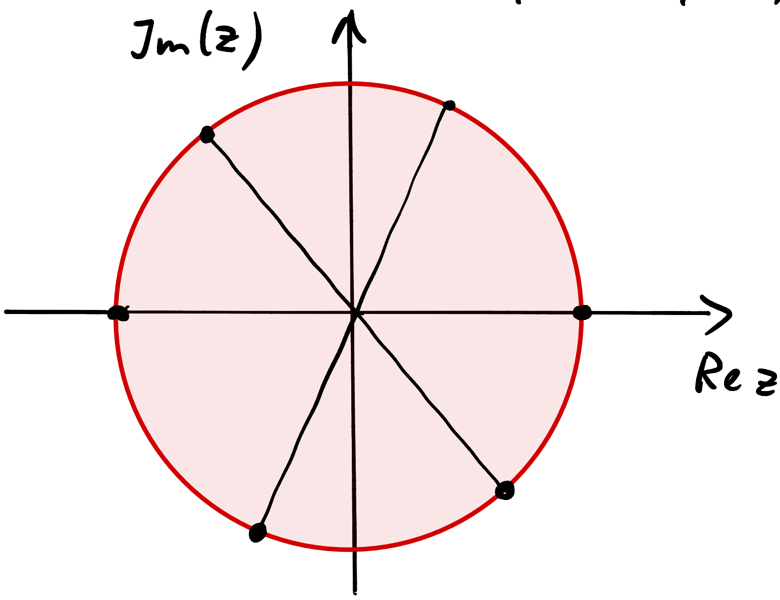
$$\sqrt[n]{z} = \left[r\, e^{i(\varphi + 2\pi k)}\right]^{\frac{1}{n}} = \sqrt[n]{r}\, e^{i\left(\frac{\varphi}{n} + \frac{k}{n} 2\pi\right)}$$

There are exactly n different n-th roots of a complex number $z = re^{i\varphi}$ with $z \neq 0$

$$w_k = \sqrt[n]{r}\, e^{i\left(\frac{\varphi}{n} + \frac{k}{n} 2\pi\right)}$$

with $k = 0, 1, \ldots, n-1$

6-th root of 1:



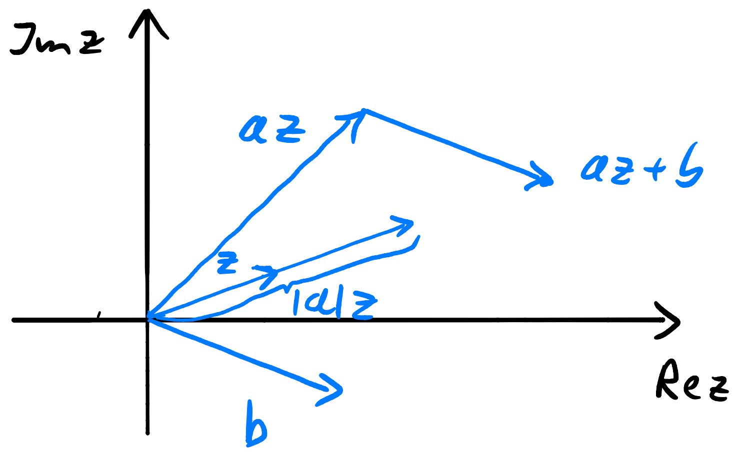
1.2 Complex Functions

Let $A \subset \mathbb{C}$ $\quad f: A \to \mathbb{C}$ is a complex function

Example (1) $\quad A = \mathbb{C}$, $f(z) = a \cdot z + b$; $\quad a, b \in \mathbb{C}$, $a \neq 0$

linear map

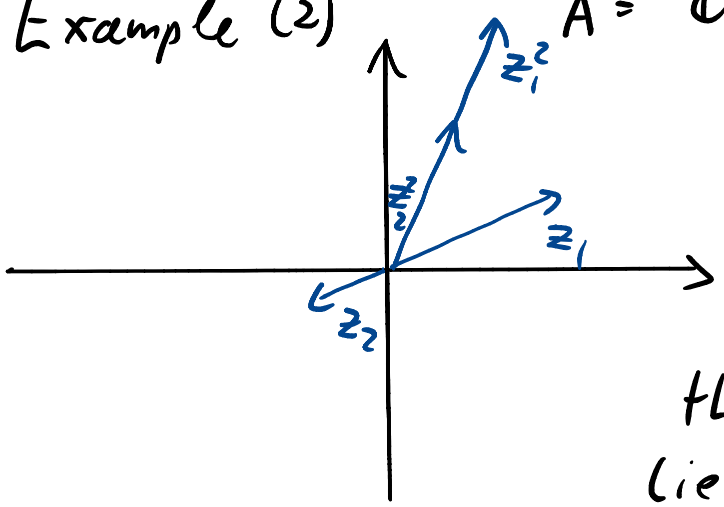
$$f: z \xmapsto{\text{stretching}} |a| z \xmapsto{\text{rotation}} az \xmapsto{\text{translation}} az + b$$



Linear maps are bijective:

$w = f(z) \qquad z = \frac{1}{a} w - \frac{b}{a}$

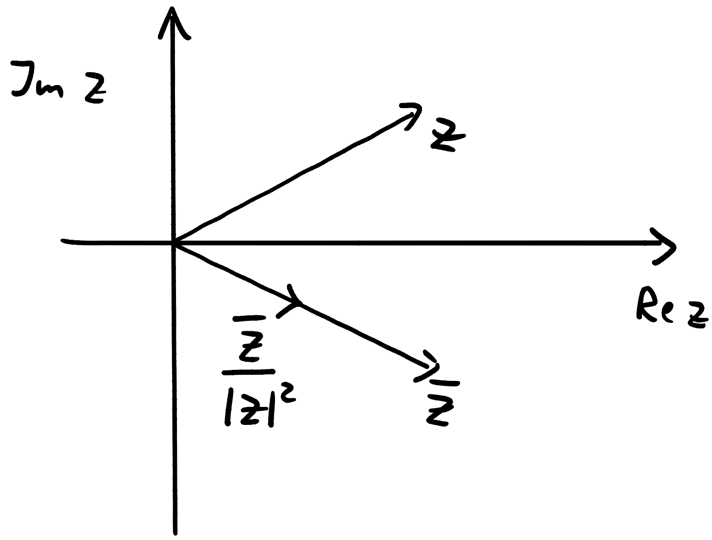
Example (2) $\quad A = \mathbb{C} \quad f(z) = z^2$



If z_1, z_2 lie on a straight line through the origin, then z_1^2 and z_2^2 lie on the same half line

Example (3) $A = \mathbb{C} \setminus \{0\}$ $\quad f(z) = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$

$$= \frac{1}{|z|}\left(\frac{\bar{z}}{|z|}\right)$$



Claim: $f(z) = \frac{1}{z}$ maps "circles" onto "circles"

("circles" = circles + straight lines)

Proof: Consider $M = \{ z = x+iy : \alpha(x^2+y^2) + \beta x + \gamma y + \delta = 0 \}$

For $\alpha = 0$: straight lines

$\alpha \neq 0$: circles

$$x^2 + y^2 + \frac{\beta}{\alpha}x + \frac{\gamma}{\alpha}y + \frac{\delta}{\alpha} = 0$$

$$\left(x + \frac{\beta}{2\alpha}\right)^2 + \left(y + \frac{\gamma}{2\alpha}\right)^2 - \frac{1}{4}\frac{\beta^2}{\alpha^2} - \frac{1}{4}\frac{\gamma^2}{\alpha^2} + \frac{\delta}{\alpha} = 0$$

Circle centered at $\left(-\frac{\beta}{2\alpha}, -\frac{\gamma}{2\alpha}\right)$ and with

radius $\sqrt{\frac{\beta^2}{4\alpha^2} + \frac{\gamma^2}{4\alpha^2} - \frac{\delta}{\alpha}}$

Now let $0 \neq z \in M$

$$\Rightarrow \quad \alpha + \beta \frac{x}{x^2+y^2} + \gamma \frac{y}{x^2+y^2} + \frac{\delta}{x^2+y^2} = 0$$

$$\Rightarrow \quad \alpha + \beta \operatorname{Re} f(z) - \gamma \operatorname{Im} f(z) + \delta |f(z)|^2 = 0$$

For $z \in M$ satisfies $f(z) = u + iv$ the equation

$$\alpha + \beta u - \gamma v + \delta (u^2+v^2) = 0 \qquad \text{circle}$$

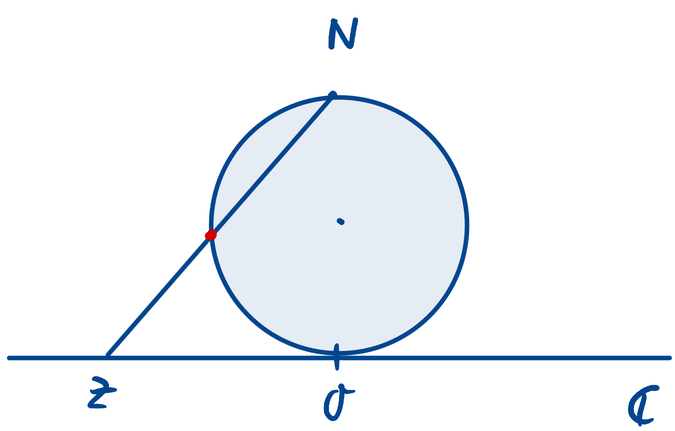
$f: \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is bijective with the inverse map $f^{-1}(w) = \frac{1}{w}$

Def.: $\tilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called extended complex plane.

Then $\tilde{f}: \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$, $\tilde{f}(z) = \begin{cases} \frac{1}{z}, & z \in \tilde{\mathbb{C}} \setminus \{\infty, 0\} \\ \infty & z = 0 \\ 0 & z = \infty \end{cases}$

Stereographic projection and Riemann sphere:

Drawing a straight line from the north pole to the number z uniquely determines a point on the sphere.



Example 4: Linear fractional transformation

$$f(z) = \frac{az+b}{cz+d} \qquad a, b, c, d \in \mathbb{C}$$

$A = \mathbb{C} \setminus \{-\frac{d}{c}\}$, assume $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$

$c = 0$ Linear map

$c \neq 0$ Claim: $f = f_1 \circ f_2 \circ f_3$

where f_1 and f_3 are linear maps, and $f_2 = \frac{1}{z}$

Proof: $f(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) - \frac{a}{c}d + b}{cz+d}$

$$= \frac{a}{c} + \frac{1}{c}\,\frac{bc-ad}{cz+d}$$

$f_3(z) = cz+d$, $f_2(z) = \frac{1}{z}$, $f_1(z) = \frac{bc-ad}{c} z + \frac{a}{c}$

With this $f(z) = f_1(f_2(f_3(z)))$

Every function $f(z) = \frac{az+b}{cz+d}$ with $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$

can be extended to a bijective map $\hat{f}$ in $\hat{\mathbb{C}}$,

which maps "circles" into "circles".

i) $c = 0$ $\hat{f}(\infty) = \infty$

ii) $c \neq 0$ $\quad \tilde{f}(\infty) = \frac{a}{c}$, $\quad \tilde{f}(-\frac{d}{c}) = \infty$

Example 5: complex exponential function

$$e^{iy} = \cos y + i \sin y$$

Def.: $z = x + iy$, then $e^z := e^x(\cos y + i \sin y)$

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(x+iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k (iy)^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\cancel{n!}\,1}{(n-k)!\,k!\,\cancel{n!}\,1} x^k (iy)^{n-k}$$

$$= \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \left(\sum_{n=0}^{\infty} \frac{(iy)^n}{n!} \right) = e^x e^{iy} = e^x(\cos y + i \sin y)$$

The exponential function is periodic with period $2\pi i$

$$\boxed{\exp(z) = \exp(z + 2\pi i)}$$

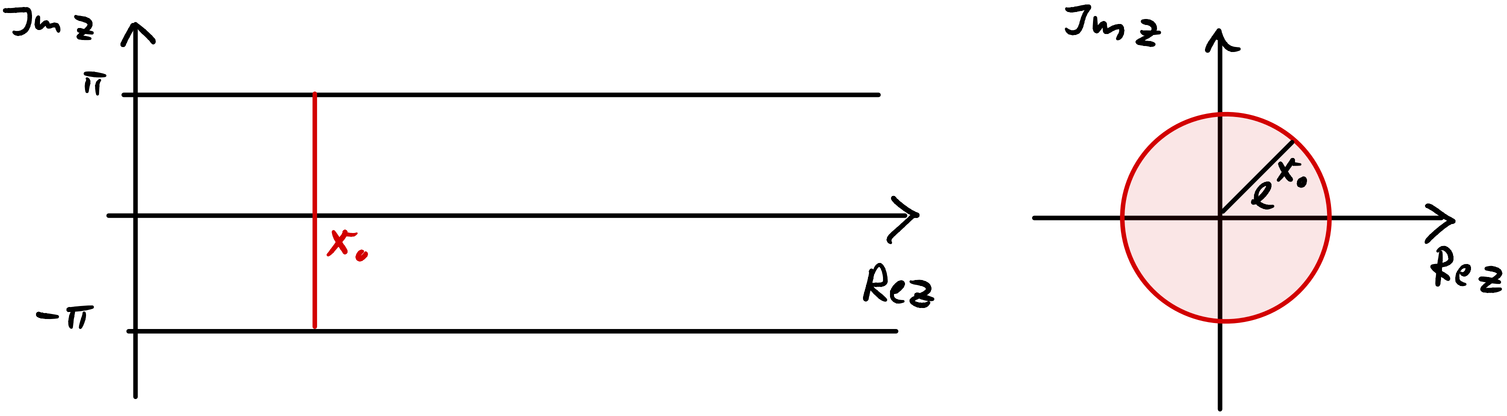
Proof: $\exp(z + 2\pi i) = \exp(x + i(y + 2\pi))$

$$= e^x[\cos(y+2\pi) + i \sin(y+2\pi)]$$

$$= \exp(z)$$

Consider $S = \{z : x+iy \mid -\pi < y \leq \pi\}$

$$e^{x_0}(\cos \varphi + i \sin \varphi) \qquad -\pi < \varphi \leq \pi$$



The straight line $x = x_0$ is mapped onto a circle with radius e^{x_0} $\Rightarrow$ $\exp: S \to \mathbb{C} \setminus \{0\}$ is bijective.

1.3 Convergence and Continuity

The norm $|z| = \sqrt{z\bar{z}} = \sqrt{x^2+y^2}$ is the Euclidean norm in $\mathbb{R}^2$.

Def.: a sequence $\{z_n\}$, $z_n \in \mathbb{C}$, $n \in \mathbb{N}$ is called convergent to z_0 $(z_n \to z_0)$, if

$$\forall_{\varepsilon>0} \ \exists_{N\in\mathbb{N}} \ \forall_{n>N} \quad |z_n - z_0| < \varepsilon$$

$$\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \qquad z_n \to \infty \overset{\text{Def}}{\Longleftrightarrow} \forall_{\varepsilon>0} \ \exists_{N\in\mathbb{N}} \ \forall_{n\geq N} \ |z_n| > \frac{1}{\varepsilon}$$

$$\Longleftrightarrow \frac{1}{z_n} \to 0$$

Def.: $c \in \mathbb{C}$ is called limit value of a complex function f at z_0 ($\lim\limits_{z \to z_0} f(z) = c$) if and onl if every sequence $\{z_n\}$ with $z_n \to z_0$ satisfies $f(z_n) \to c$.

Def.: Let $A \subseteq \tilde{\mathbb{C}}$, $f: A \to \mathbb{C}$

f is called continuous at $z_0 \in A$ $\Leftrightarrow$

$\lim\limits_{z \to z_0} f(z) = f(z_0)$ $\Leftrightarrow$

$\{z_n\} \subset A$, $z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$

$$\Leftrightarrow \forall_{\varepsilon > 0} \; \exists_{\delta > 0} \; \forall_{|z - z_0| < \delta} \quad |f(z) - f(z_0)| < \varepsilon$$

Example ⑥ Polynomials $f(z) = \sum\limits_{j=0}^{n} a_j z^j$ are continuous in $\mathbb{C}$, $a_j \in \mathbb{C}$

Example ⑦ rational functions $f(z) = \dfrac{p(z)}{q(z)}$

p, q polynomials, $A = \{z \in \mathbb{C} : q(z) \neq 0\}$ are continuous in A.

Example ⑧ Logarithm

Def.: Let $A = \mathbb{C} \setminus \{0\}$ $\quad \ln z := \ln|z| + i \arg z$

1) For real z this definition agrees with the usual definition of the logarithm.

2) $\ln z$ is the reverse function of e^z

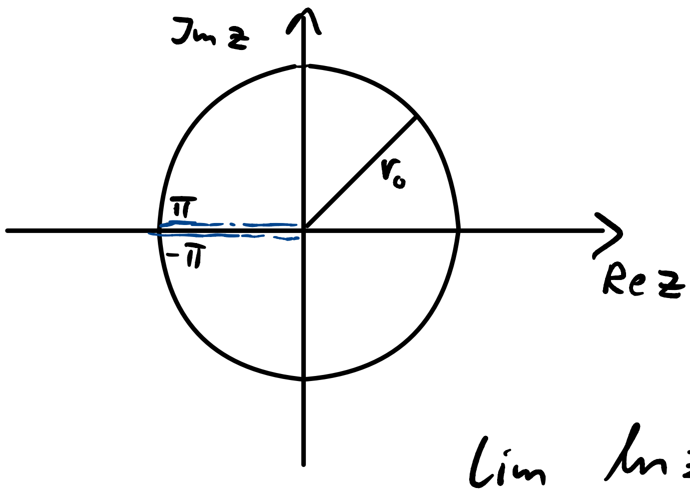
Proof: Let $z = x + iy$

$$w = e^z = e^x e^{iy}$$

$$\ln w = \ln|w| + i \underbrace{\arg w}_{y} = \ln e^x + iy = x + iy$$

Problem: $\arg z$ is not unique

$\rightarrow$ restrict $\arg z$ to $(-\pi, \pi)$



$$z = r_0 e^{i\varphi}$$

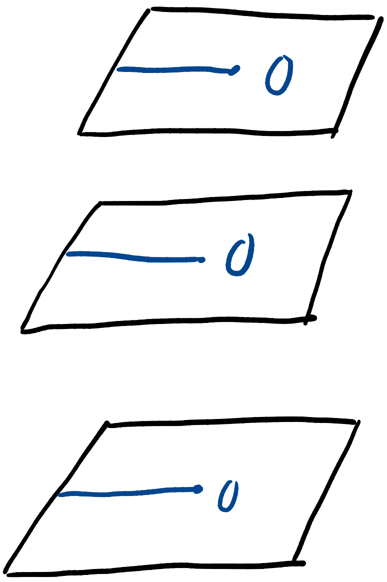
$$\lim_{\varphi \to \pi} \ln z = \lim_{\varphi \to \pi} (\ln r_0 + i\varphi) = \ln r_0 + i\pi$$

$$\lim_{\varphi \to -\pi} \ln z = \lim_{\varphi \to -\pi} (\ln r_0 + i\varphi) = \ln r_0 - i\pi$$

but $\lim_{\varphi \to \pi} z = -r_0 \qquad \lim_{\varphi \to -\pi} z = -r_0$

Riemann sheets

Expand the definition of arg z such that the logarithm is continuous



$\pi < \arg z \leq 3\pi$

$-\pi < \arg z \leq \pi$

$-3\pi < \arg z \leq -\pi$

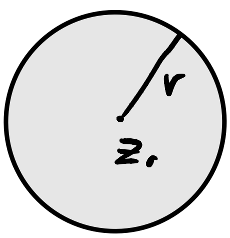
move from one sheet to the other when approaching the 'cut'

Other example for discontinuity: $f(z) = \sqrt[n]{r}\, e^{i\varphi/n}$

1.4 Complex Differentiation

Def.: $K(z_0, r) := \{ z \in \mathbb{C} : |z - z_0| < r \}$

is the "open ball" around z_0 with radius r.

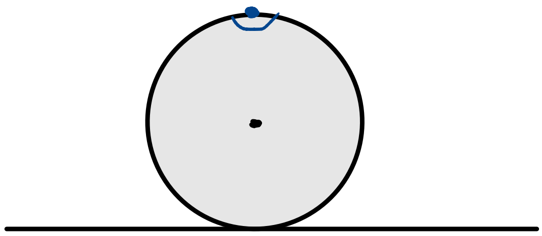


area of the circle without the boundary.

Def.: Let $A \subset \mathbb{C}$. A $z \in \mathbb{C}$ is called inner point of A if there exists an ε such that $K(z, \varepsilon) \subset A$.

Def.: $\Omega \subset \mathbb{C}$ is called open if all its points are inner points. $A \subset \mathbb{C}$ is called closed if $\mathbb{C} \setminus A$ is open.

Extension to $\tilde{\mathbb{C}}$



$$K(\infty, \varepsilon) := \left\{ z \in \mathbb{C} : |z| > \frac{1}{\varepsilon} \right\} \cup \{\infty\}$$

Def.: Let $\Omega \subset \mathbb{C}$ be open, $z_0 \in \Omega$ and $f : \Omega \to \mathbb{C}$.

f is called partially differentiable w.r.t. x or y in z_0, if the following limits exist:

$$f_x(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h} \left[f(x_0 + h + iy_0) - f(x_0 + iy_0) \right]$$

$$f_y(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h} \left[f(x_0 + i(h + y_0)) - f(x_0 + iy_0) \right]$$

f is called continuously partially differentiable in Ω

if $f_x(z)$ and $f_y(z)$ exist for all $z \in \Omega$ and are continuous.

Def.: Let $\Omega \in \mathbb{C}$ be open, $f: \Omega \to \mathbb{C}$, $z_0 \in \Omega$

f is called complex differentiable at z_0 if the limit $f'(z_0) = \lim\limits_{\substack{z \to z_0 \\ z \neq z_0}} \frac{f(z)-f(z_0)}{z - z_0}$ exists.

Theorem: f is complex differentiable in z_0
$\Rightarrow f$ is continuous at z_0

Proof.: f complex differentiable $\Rightarrow f'(z_0)$ exists.

$$\Rightarrow \varphi(z) := \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \xrightarrow[z \to z_0]{} 0$$

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \varphi(z)(z-z_0)$$

$$\xrightarrow[z \to z_0]{} f(z_0)$$

Theorem: Let f be complex differentiable at z_0. Then the partial derivatives of f exist in z_0, and the following holds

$$f'(z_0) = f_x(z_0) = \frac{1}{i} f_y(z_0)$$

Remark: When letting $f = u + iv$ with u, v real-valued functions, then the Cauchy-Riemann differential equations

$$u_x = v_y \quad , \quad v_x = -u_y$$

are equivalent to $f_x = \frac{1}{i} f_y$

Proof: Consider to special sequences

1) $z := z_0 + h$ with $h \in \mathbb{R}$, $h \to 0$

$$f'(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h} \left[f(z_0 + h) - f(z_0) \right]$$

$$= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h} \left[f(x_0 + h + iy_0) - f(x_0 + iy_0) \right]$$

$$= f_x(z_0)$$

2) $z := z_0 + ih \qquad h \in \mathbb{R},\ h \to 0$

$$f'(z_0) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{1}{ih}\left[f(z_0+ih) - f(z_0)\right]$$

$$= \frac{1}{i} \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{1}{h}\left[f(x_0 + i(h+y_0)) - f(x_0+iy_0)\right]$$

$$= \frac{1}{i} f_y(z_0)$$

Proof of the remark: $\quad f(z_0) = u(z_0) + i v(z_0)$

$$\Rightarrow \quad f_x(z_0) = u_x(z_0) + i v_x(z_0)$$

$$f_y(z_0) = u_y(z_0) + i v_y(z_0)$$

$$\frac{1}{i} f_y(z_0) = v_y(z_0) - i u_y(z_0)$$

Hence, $\quad f_x = \frac{1}{i} f_y \iff u_x = v_y \ ,\ v_x = -u_y$

Theorem: Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ be continuously partially differentiable at $z_0 \in \Omega$, and let $f_x(z_0) = \frac{1}{i} f_y(z_0)$. Then f is complex differentiable at z_0.

Remark: Since $f'(z_0) = f_x(z_0)$, f is even continuously differentiable at z_0.

Proof: Use a relation from real analysis:

Let $\Omega \in \mathbb{R}^n$ be open, $f : \Omega \to \mathbb{R}^m$ be continuously partially differentiable at $z_0 \in \Omega$. Then there exists a function $\varphi : \Omega \to \mathbb{R}^m$ with $\varphi(z_0) = 0$, such that the following relation hold ($(Df)(z_0)$ denotes the Jacobian matrix of partial derivatives)

$$f(z) = f(z_0) + (Df)(z_0) \cdot (z - z_0) + (z - z_0) \cdot \varphi(z)$$

Identify the complex function $f : \Omega \to \mathbb{C}$ with

$$f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad z = \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u(z) \\ v(z) \end{pmatrix} = f(z)$$

There exists φ such that

$$f(z) - f(z_0) = (Df)(z - z_0) + (z - z_0)\varphi(z) \quad (*)$$

$$(Df)(z_0)(z - z_0) = \begin{bmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

$$= \begin{bmatrix} u_x(x - x_0) + u_y(y - y_0) \\ v_x(x - x_0) + v_y(y - y_0) \end{bmatrix}$$

$$\overset{CR}{=} \begin{bmatrix} u_x(x-x_0) - v_x(y-y_0) \\ v_x(x-x_0) + u_x(y-y_0) \end{bmatrix}$$

$$= u_x(z_0)\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + v_x(z_0)\begin{bmatrix} -(y-y_0) \\ x-x_0 \end{bmatrix}$$

$$= \left[u_x(z_0) + i v_x(z_0)\right](z-z_0)$$

Divide $(*)$ by $(z-z_0)$

$$\frac{f(z)-f(z_0)}{z-z_0} = u_x(z_0) + i v_x(z_0) + \underbrace{\frac{z-z_0}{z-z_0} f(z)}_{\xrightarrow[z \to z]{} 0}$$

$$\Big\downarrow z \to z_0$$

$$f'(z_0) = u_x(z_0) + i v_y(z_0) = f_x(z_0)$$

$\Rightarrow$ f is complex differentiable at z_0

Def.: Let $\Omega \subset \mathbb{C}$ be open. $f: \Omega \to \mathbb{C}$ is called holomorphic if f is complex differentiable at all $z \in \Omega$ and f' is continuous in Ω.

f is holomorphic at ∞ if $g(z) := f\left(\frac{1}{z}\right)$ is holomorphic at 0.

Remark: If f is holomorphic in a disc around z_0, then f is called holomorphic at z_0.

Example ①: $f: \mathbb{C} \to \mathbb{C}$, $f(z) = z$ (identity)

$$f_x(z) = 1, \quad f_y(z) = i \quad \forall_{z \in \mathbb{C}}$$

f_x and f_y are continuous and we have $f_x = \frac{1}{i} f_y$
$\Rightarrow f$ is holomorphic in $\mathbb{C}$.

② $f(z) = \frac{1}{z}$ is holomorphic at ∞, since
$g(z) := f\left(\frac{1}{z}\right) = z$ is holomorphic at 0.

③ $f: \mathbb{C} \to \mathbb{C}$ $\quad f(z) = Re(z)$

f is partially differentiable $f_x(z) = 1$, $f_y(z) = 0$
$\Rightarrow f_x \neq \frac{1}{i} f_y \Rightarrow f(z)$ is not complex differentiable

Theorem: Let $\Omega \subset \mathbb{C}$ be open, $u: \Omega \to \mathbb{R}$ two times continuously partial differentiable.
Then the following holds:

$$\Delta u := u_{xx} + u_{yy} = 0 \iff \exists \text{ a function } f \text{ holomorphic in } \Omega \text{ with } Re(f) = u$$

Proof: "$\Leftarrow$" Let $f = u + iv$ be holomorphic

$\Rightarrow$ $u_x = v_y$, $u_y = -v_x$

$\Rightarrow$ $u_{xx} = v_{yx}$, $u_{yy} = -v_{xy}$

$$\left.\begin{array}{l} u_{xx} \text{ continuous} \Rightarrow v_{yx} \text{ continuous} \\ u_{yy} \text{ continuous} \Rightarrow v_{xy} \text{ continuous} \end{array}\right\}$$

$\Rightarrow$ $v_{yx} = v_{xy}$

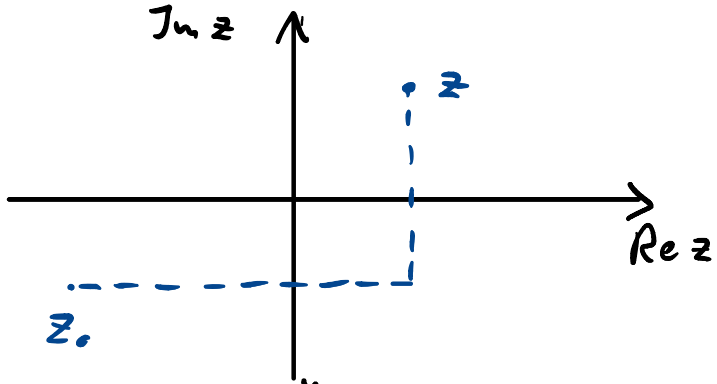
$\Rightarrow$ $\Delta u = u_{xx} + u_{yy} = v_{yx} - v_{yx} = 0$

"$\Rightarrow$" Let $u : \Omega \to \mathbb{R}$ be given, with $\Delta u = 0$

We need to find $v : \Omega \to \mathbb{R}$, such that $f = u + iv$ is holomorphic, i.e. $v_x = -u_y$, $v_y = u_x$

Choose a $z_0 \in \Omega$ and let $v(z_0) = 0$

Determine $v(z)$ via integration over the following path



$$v(z) := \int_{x_0}^{x} v_x(t, y_0)\,dt + \int_{y_0}^{y} v_y(x, s)\,ds$$

$$= -\int_{x_0}^{x} u_y(t, y_0)\,dt + \int_{y_0}^{y} u_x(x, s)\,ds$$

Check that the Cauchy–Riemann differential eqs. are satisfied:

$$v_x(x,y) = -u_y(x,y_0) + \int_{y_0}^{y} u_{xx}(x,s)\,ds$$

$$= -u_y(x,y_0) - \int_{y_0}^{y} u_{yy}(x,s)\,ds$$

$$= -u_y(x,y_0) - u_y(x,y) + u_y(x,y_0) = -u_y(x,y)$$

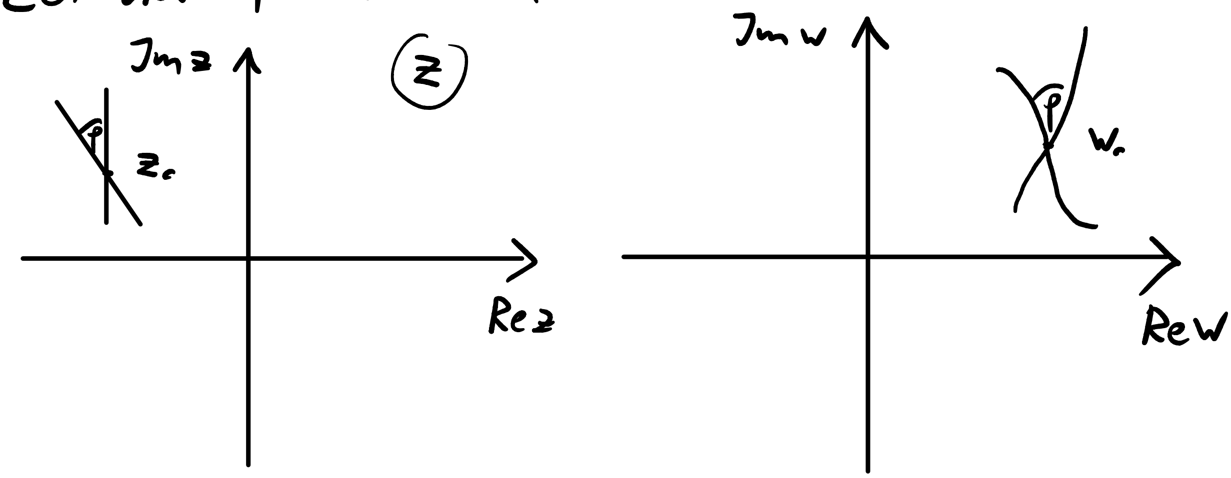
Geometrical interpretation of complex differentiability

Let f be holomorphic at z_0.

$$\rho(z) := \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \xrightarrow[z \to z_0]{} 0$$

$$\Rightarrow f(z) = f(z_0) + f'(z_0)(z-z_0) + \rho(z)(z-z_0)$$

Let now $f'(z_0) \neq 0$, $w = f(z)$, $w_0 = f(z_0)$



Straight line through z_0 $\qquad z(t) = z_0 t e^{i\alpha}$ $t \in \mathbb{R}$

$$f'(z_0) = a = |a|\, e^{i \arg a}$$

$$w(t) = f(z(t)) = f(z_0) + f'(z_0)te^{i\alpha} + \underbrace{\rho(z_0+te^{i\alpha})te^{i\alpha}}_{\to 0 \text{ for } t\to 0}$$

$$\approx w_0 + a t e^{i\alpha} = w_0 + |a| t\, e^{i(\alpha + \arg a)}$$

Consider two curves, which intersect each other at a point z_0. These curves are mapped onto two curves in the w-plane, which intersect each other at w_0 with the same angle as the original curves in the z-plane.

Rules for differentiation:

1. f, g are holomorphic at z_0, and $a, b \in \mathbb{C}$
   $\Rightarrow$ $af + bg$, $f \cdot g$, $\frac{f}{g}$ $(g(z_0) \neq 0)$ are holomorphic as well and one has

   $(af+bg)' = af' + bg'$

   $(f \cdot g)' = f'g + fg'$

   $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$

2. Chain rule

   f holomorphic at z_0, F holomorphic at $f(z_0)$

   $\Rightarrow$ $g = F \circ f$ holomorphic at z_0, and one has

   $$g'(z_0) = F'(f(z_0)) \cdot f'(z_0)$$

Examples: ① $f(z) = z^n$ proof by induction

f holomorphic for all $n \in \mathbb{N}$, and $f'(z) = n z^{n-1}$

② Polynomials: $f(z) = a_0 + a_1 z + \dots + a_n z^n$

holomorphic in $\mathbb{C}$ and

$f'(z) = a_1 + 2a_2 z + \dots + n a_n z^{n-1}$

③ rational functions $f(z) = \dfrac{p(z)}{q(z)}$ are holomorphic

in $A = \{ z \in \mathbb{C} : q(z) \neq 0 \}$

Let $p(z) = \sum_{j=0}^{m} a_j z^j$, $q(z) = \sum_{j=0}^{n} b_j z^j$, $m \leq n$

Extension of the domain:

$$\tilde{A} = \{ z \in \tilde{\mathbb{C}} : q(z) \neq 0 \} \text{ and let}$$

$$f(\infty) = \begin{cases} \dfrac{a_m}{b_m} & m = n \\ 0 & m < n \end{cases}$$

$$f(z) = \frac{a_0 + a_1 z + \dots + a_m z^m}{b_0 + b_1 z + \dots + b_n z^n} = z^{m-n} \frac{\frac{a_0}{z^m} + \frac{a_1}{z^{m-1}} + \dots + a_m}{\frac{b_0}{z^n} + \frac{b_1}{z^{n-1}} + \dots + b_n}$$

$$\xrightarrow[z \to \infty]{} \begin{cases} \dfrac{a_m}{b_n} & m = n \\ 0 & m < n \end{cases}$$

Claim: f is holomorphic at ∞ (for $n \geq m$)

Proof: we need to show that $g(z) := f(\frac{1}{z})$ is holomorphic at 0.

$$g(z) = f\left(\tfrac{1}{z}\right) = z^{n-m} \frac{a_0 z^m + a_1 z^{m-1} + \dots + a_m}{b_0 z^n + b_1 z^{n-1} + \dots + b_n}$$

The denominator is different from zero for $z = 0$

$\Rightarrow$ g holomorphic at 0, since z^{n-m} holomorphic at 0 due to $n \geq m$ $\Rightarrow$ fraction is holomorphic as well.

④ Exponential function

Claim: $f(z) = \exp(z)$ is holomorphic for $\forall_{z \in \mathbb{C}}$, and $(e^z)' = e^z$

Proof: $\exp(x+iy) = e^x(\cos y + i \sin y)$

$f_x = e^x(\cos y + i \sin y)$, $\frac{1}{i} f_y = \frac{1}{i} e^x(-\sin y + i \cos y)$

$= e^x(\cos y + i \sin y)$

$\Rightarrow f_x = \frac{1}{i} f_y$, f_x is continuous everywhere

$\Rightarrow f$ is holomorphic.

$f'(z) = f_x(z) = \exp(z)$

⑤ $f(z) = \bar{z} = x - iy$

$f_x = 1 \neq \frac{1}{i} f_y = \frac{-i}{i} = -1 \qquad \forall_{z \in \mathbb{C}}$

⑥ The real valued function $f: \mathbb{R} \to \mathbb{R}$

$f(x) = \begin{cases} 0 & : x \leq 0 \\ x^2 & : x > 0 \end{cases}$ continuously differentiable on the real axis

Interpret f as a complex function

$$f: \mathbb{R} =: \Omega \to \mathbb{C}$$

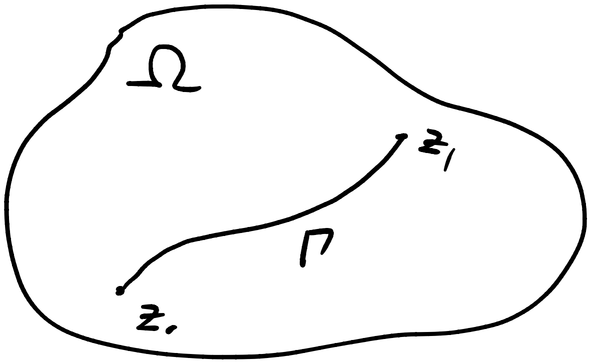
$$f_x = 2x, \quad \frac{1}{i} f_y = \frac{1}{i} \cdot 0 = 0$$

$f_x \neq \frac{1}{i} f_y$ for $x > 0$, f is not complex differentiable

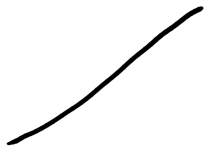
Since $\mathbb{R}$ is not open in $\mathbb{C}$, the definition of differentiability is difficult, holomorphy is impossible to define.

§2 Integral Theorems

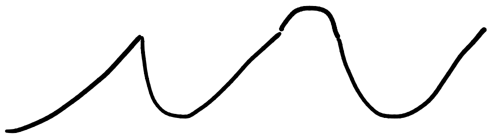
2.1 Line Integrals



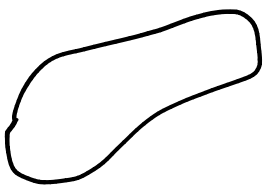
"rectifiable curve"



path segment



path as a combination of segments



curve

Def.: $\gamma : [a,b] \to \mathbb{C}$ continuously differentiable

(i.e. $\gamma = \gamma_1 + i\gamma_2$, $\gamma_j : [a,b] \to \mathbb{R}$ continuously differentiable)

and the following holds true:

1) $\varphi'(t) = \varphi_1'(t) + i\,\varphi_2'(t) \neq 0 \quad \forall_{t \in [a,b]}$

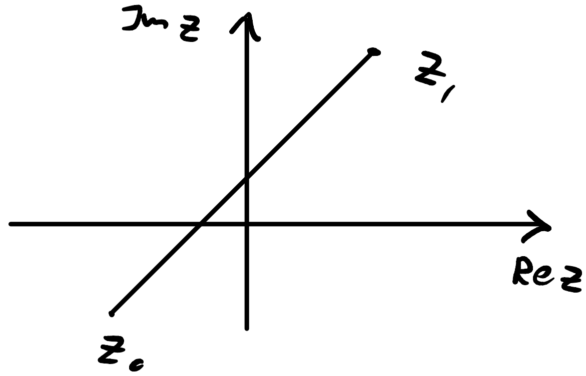
2) $\varphi(t) \neq \varphi(\tilde{t})$ with $t \neq \tilde{t}$

Then $\Gamma(\varphi(a), \varphi(b)) = \{ z \in \mathbb{C} : z = \varphi(t),\ a \leq t \leq b \}$ is called path segment in $\mathbb{C}$ with initial point $z_0 = \varphi(a)$ and final point $z_1 = \varphi(b)$. $\varphi(t)$ is called parametrization of the path segment.

Def.: $-\Gamma = \{ z \in \mathbb{C} : z = \varphi(-t),\ -b \leq t \leq -a \}$ is called the inverse path segment.

$\psi : [-b, -a] \to \mathbb{C}$, $\psi(t) = \varphi(-t)$ is the parametrization of $-\Gamma$

Example:



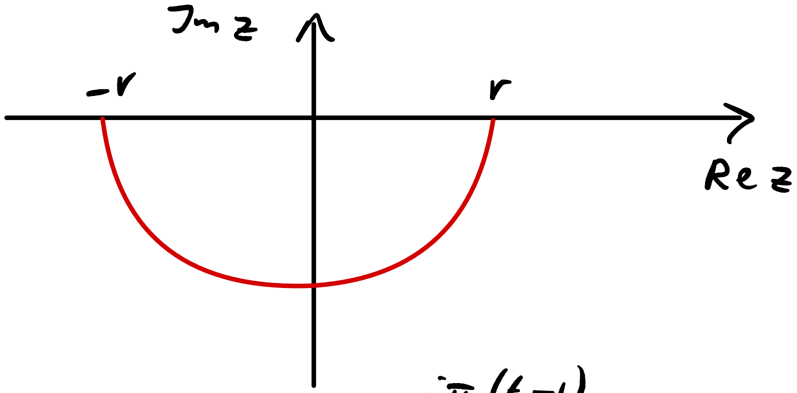
$$\varphi_1(t) = (1-t)\,z_0 + t\,z_1 \,, \quad \varphi_1 : [0,1] \to \mathbb{C}$$

$$\varphi_2(t) = z_0 + \frac{t}{|z_1 - z_2|}(z_1 - z_0) \,, \quad \varphi_2 : [0, |z_1 - z_2|] \to \mathbb{C}$$

$$\varphi_3(t) = z_0 \left(\cos \tfrac{\pi}{2} t\right)^2 + z_1 \left(\sin \tfrac{\pi}{2} t\right)^2 , \quad \varphi_3 : [0,1] \to \mathbb{C}$$

$-\Gamma : \psi(t) = (1-t) z_1 + t z_0 \qquad \psi : [0,1] \to \mathbb{C}$

Example semi-circle

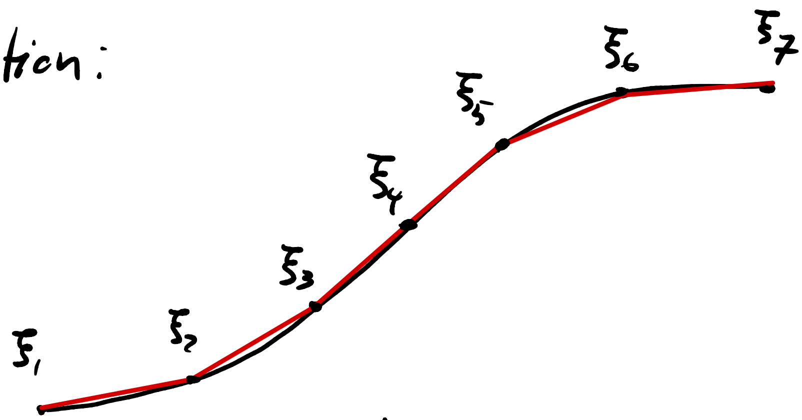


parametrization $\varphi(t) = r e^{i\pi(t-1)}, \quad \varphi : [0,1] \to \mathbb{C}$

inverse path segment : $\psi(t) = r e^{i\pi(-t)}, \quad \psi : [0,1] \to \mathbb{C}$

Let $\varphi : [a,b] \to \mathbb{C}$ be a parametrization of Γ,
then $|\Gamma| = \int_a^b |\varphi'(t)| dt$ is the length of Γ.

Motivation:



$$|\Gamma| = \sum_{j=2}^{N} |\xi_j - \xi_{j-1}| \qquad \xi_j = \varphi(t_j)$$

$$= \sum_{j=2}^{N} \frac{|\varphi(t_j) - \varphi(t_{j-1})|}{t_j - t_{j-1}} (t_j - t_{j-1})$$

$$\approx \sum_{j=2}^{N} |\varphi'(t_j)| \, (t_j - t_{j-1})$$

$$\xrightarrow[N \to \infty]{} \int_a^b |\varphi'(t)| \, dt$$

Lemma: Let $\varphi : [a,b] \to \mathbb{C}$, $\psi : [c,d] \to \mathbb{C}$

be parametrizations of the same path segment Γ.

Then there exists a continuously diff. function

$$\tau : [c,d] \to [a,b]$$

with $\tau(c) = a$, $\tau(d) = b$ and $\tau'(t) > 0$ for $t \in (c,d)$,

such that $\psi(t) = \varphi(\tau(t))$

Proof: $\varphi : [a,b] \to \Gamma$ is bijective

Def.: $\tau := \varphi^{-1} \circ \psi$

$$\Rightarrow \tau(c) = \varphi^{-1}(\psi(c)) = \varphi^{-1}(\varphi(a)) = a$$

$$\tau(d) = \varphi^{-1}(\psi(d)) = \varphi^{-1}(\varphi(b)) = b$$

φ, ψ are continuous and bijective $\Rightarrow$ τ is continuous, bijective

$\Rightarrow \tau'(t) \neq 0$ for $t \in (c,d)$

in addition $\tau'(t) > 0$, since $\tau(c) = a < b = \tau(d)$

chain rule, differentiation of the inverse function

$$\tau'(t) = \frac{d}{dt}\left[\varphi^{-1}(\psi(t))\right] \qquad\qquad [f^{-1}(y)]' = \frac{1}{f(x)}$$

$$= \frac{d}{d\psi}\varphi^{-1}(\psi) \quad \frac{d\psi}{dt}$$

$$= \frac{1}{\varphi'(\tau(t))} \quad \psi'(t)$$

Theorem: Let $\varphi : [a,b] \to \mathbb{C}$ and $\psi : [c,d] \to \mathbb{C}$

be parametrizations of a path segment Γ.

Then $|\Gamma| = \int_a^b |\varphi'(t)|\,dt = \int_c^d |\psi'(t)|\,dt$

Proof: $\psi(t) := \varphi(\tau(t))$ with $\tau(t)$ from above.

$$\Rightarrow \psi'(t) = \varphi'(\tau(t))\,\tau'(t) \quad , \tau'(t) > 0$$

$$\Rightarrow \int_c^d |\psi'(t)|\,dt = \int_c^d |\varphi'(\tau(t))|\,\tau'(t)\,dt$$

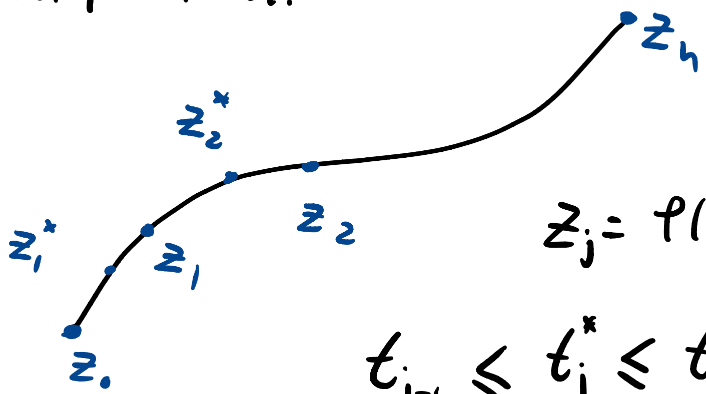
substitute $s := \tau(t)$

$$= \int_{\tau(c)}^{\tau(d)} |\varphi'(s)|\,ds = \int_a^b |\varphi'(s)|\,ds \qquad \#$$

Def.: Let Γ be a path segment and $f: \Gamma \to \mathbb{C}$ continuous, and φ is a parametrization of Γ. Then the integral of f along Γ is defined as

$$\boxed{\int_\Gamma f(z)\,dz := \int_a^b f(\varphi(t))\,\varphi'(t)\,dt}$$

Justification:



$$z_j = \varphi(t_j), \quad z_j^* = \varphi(t_j^*)$$

$$t_{j-1} \leq t_j^* \leq t_j$$

$$\int_\Gamma f(z)\,dz \approx \sum_{j=1}^n f(z_j^*)(z_j - z_{j-1})$$

$$= \sum_{j=1}^n f(\varphi(t_j^*)) \underbrace{\frac{\varphi(t_j) - \varphi(t_{j-1})}{t_j - t_{j-1}}}_{\varphi'(t_j^*)} (t_j - t_{j-1})$$

$$\approx \int_a^b f(\varphi(t))\,\varphi'(t)\,dt$$

Theorem: Let $f : \Gamma \to \mathbb{C}$ be continuous, $\varphi : [a, b] \to \Gamma$, $\psi : [c, d] \to \Gamma$ are parametrizations.

Then the integral of f over Γ is independent of parametrization:

$$\int_a^b f(\varphi(t))\varphi'(t)\,dt = \int_c^d f(\varphi(t))\psi'(t)\,dt$$

Proof:

$$\int_a^b f(\varphi(t))\varphi'(t)\,dt = \int_{\tau^{-1}(a)}^{\tau^{-1}(b)} f\Big(\underbrace{\varphi(\tau(s))}_{\psi(s)}\Big)\underbrace{\varphi'(\tau(s))\tau'(s)}_{\psi'(s)}\,ds$$

substitute $t = \tau(s)$

$$= \int_c^d f(\psi(s))\psi'(s)\,ds$$

Consequences:

① Let Γ be a path segment, then $|\Gamma| = |-\Gamma|$

$$|\Gamma| = \int_a^b |\varphi'(t)|\,dt \underset{t \to -t}{=} \int_{-b}^{-a} |\varphi'(-t)|\,dt$$

$$= \int_{-b}^{-a} \left|-\frac{d}{dt}\varphi(-t)\right| dt = \int_{-b}^{-a} \left|\frac{d}{dt}\varphi(-t)\right| dt$$

② The following is true: $\int_\Gamma f(z)\,dz = -\int_{-\Gamma} f(z)\,dz$

Proof: $\int_\Gamma f(z)\,dz = \int_a^b f(\varphi(t))\varphi'(t)\,dt$

$$\underset{t\to -t}{=} \int_{-b}^{-a} f(\varphi(-t))\,\varphi'(-t)\,dt = -\int_{-b}^{-a} f(\varphi(-t))\frac{d}{dt}\varphi(-t)\,dt$$

$$= -\int_{-\Gamma} f(z)\,dz$$

③ $$\left|\int_\Gamma f(z)\,dz\right| \leq |\Gamma| \cdot \max\{|f(z)| : z \in \Gamma\}$$

Proof: $$\left|\int_\Gamma f(z)\,dz\right| = \left|\int_a^b f(\varphi(t))\,\varphi'(t)\,dt\right|$$

$$\leq \int_a^b |f(\varphi(t))| \cdot |\varphi'(t)|\,dt$$

$$\leq \max(f) \int_a^b |\varphi'(t)|\,dt$$

$$= \max(f)\,|\Gamma|$$

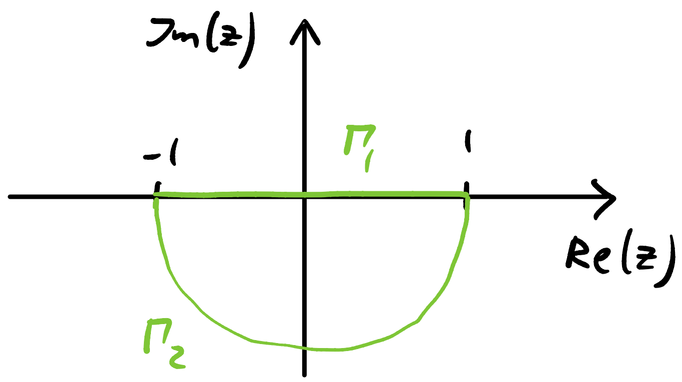
④ $$\int_\Gamma [af(z) + bg(z)]\,dz = a\int_\Gamma f(z)\,dz + b\int_\Gamma g(z)\,dz$$

⑤ Let the sequence $f_n \to f$ converge uniformly on Γ

$$\Rightarrow \int_\Gamma f_n(z)\,dz \to \int_\Gamma f(z)\,dz$$

Example: Let $f(z) = z^n$, $n \in \mathbb{N}_0$

Compute $\int_\Gamma z^n\,dz$ for



$\Gamma_1 : \quad \varphi : [-1, +1] \to \Gamma_1 \qquad \varphi(t) = t$

$\Gamma_2 : \quad \varphi : [0, 1] \to \Gamma_2 \quad , \quad \varphi(t) = e^{i\pi(t-1)}$

$$\int_{\Gamma_1} z^n \, dz = \int_{-1}^{1} t^n \cdot 1 \, dt = \frac{t^{n+1}}{n+1}\Big|_{-1}^{1} = \frac{1 - (-1)^{n+1}}{n+1}$$

$$\int_{\Gamma_2} z^n \, dz = \int_0^1 e^{ni\pi(t-1)} \, i\pi \, e^{i\pi(t-1)} \, dt$$

$$= i\pi \int_0^1 e^{(n+1)i\pi(t-1)} \, dt$$

$$= i\pi \int_0^1 \left\{ \cos\left[(n+1)\pi(t-1)\right] + i\sin\left[(n+1)\pi(t-1)\right] \right\} dt$$

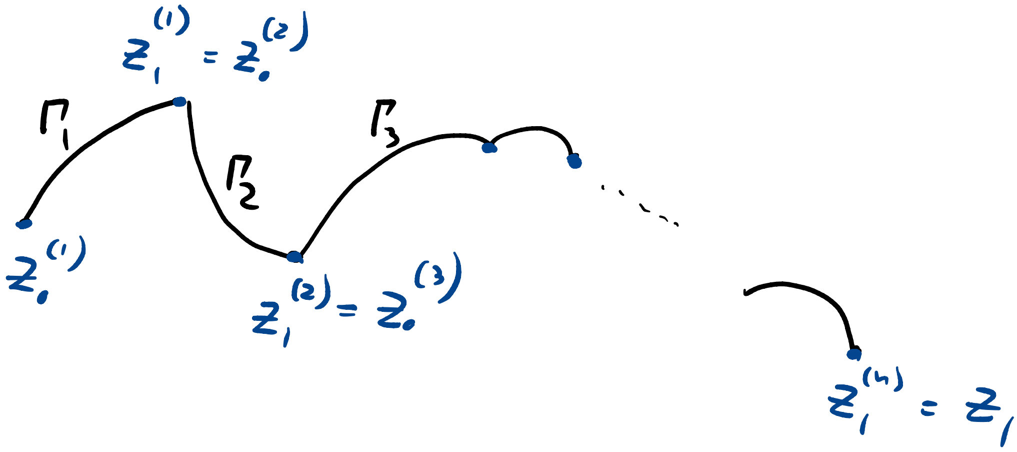
$$= \frac{i}{n+1} \left\{ \sin\left[(n+1)\pi(t-1)\right] - i\cos\left[(n+1)\pi(t-1)\right] \right\}_0^1$$

$$= \frac{i}{n+1} \left[0 - i - 0 + i(-1)^{n+1} \right]$$

$$= \frac{1 - (-1)^{n+1}}{n+1}$$ the same result a for Γ_1!

Def.: Let $\Gamma_1, \dots, \Gamma_n$ be path segments with initial points $z_0^{(j)}$ and final points $z_1^{(j)}$, $j = 1, \dots, n$

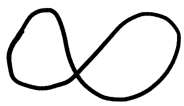
Further let $z_1^{(j-1)} = z_0^{(j)}$, $j = 2, \dots, n$



Then we call $\Gamma = \Gamma_1 + \Gamma_2 + \dots + \Gamma_n$ a contour (or more generally a path)

$z_0 = z_0^{(1)}$ is called initial point of Γ, $z_1 = z_1^{(n)}$ final point of Γ. The direction of Γ is determined by the direction of the Γ_i. Γ is called a closed contour if $z_1 = z_0$.

Γ is called simple (non-self-intersecting) if every point is reached only once when following Γ


non-simple contour

If $\Gamma = \Gamma_1 + \Gamma_2 + \dots + \Gamma_n$ is a contour, then

$$-\Gamma := (-\Gamma_n) + (-\Gamma_{n-1}) + \dots + (-\Gamma_1)$$ is

the inverse contour.

Definition: Let $\Gamma = \Gamma_1 + \dots + \Gamma_n$ be a contour, and $f: \Gamma \to \mathbb{C}$ be continuous.

Then the integral of f along Γ is defined as

$$\int_{\Gamma} f(z)dz = \sum_{j=1}^{n} \int_{\Gamma_j} f(z)dz$$

Def.: Let Γ be a contour, then $\Delta = \Gamma + (-\Gamma)$ is called null contour.

Consequences: ① $\int_{\Gamma_1 + \Gamma_2} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$

② $\int_{-\Gamma} f(z)dz = -\int_{\Gamma} f(z)dz$

③ Let $\Delta = \Gamma - \Gamma$ be the null contour, then

$$\int_{\Delta} f(z)dz = 0 \qquad \text{from ① and ②}$$

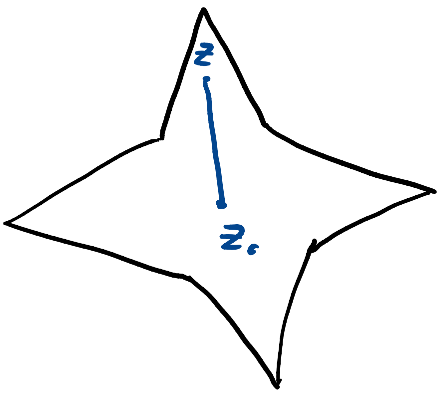
④ For integration along the real axis, contour integration is equivalent to the usual integration.

Let $\Gamma = [a,b]$, $\varphi : [a,b] \xrightarrow{id} [a,b]$

$$\Rightarrow \int_{\Gamma} f(z)dz = \int_a^b f(\varphi(t))\varphi'(t)dt = \int_a^b f(t)dt$$

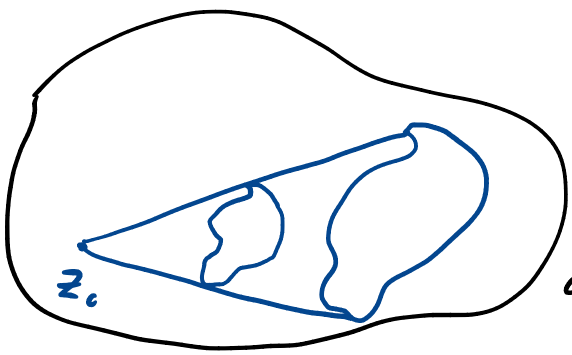
2.2 Cauchy's Integral Theorem

Def.: $\Omega \subset \mathbb{C}$ is called star-shaped if there exists a $z_0 \in \Omega$ such that for every $z \in \Omega$ the connecting line between z_0 and z lies in Ω, i.e. the set $\{(1-t)z_0 + tz : t \in [0,1]\} \subset \Omega$



Theorem: Let $\Omega \subset \mathbb{C}$ be open and star-shaped, $f : \Omega \to \mathbb{C}$ holomorphic. If $\Gamma = \Gamma_1 + \dots \Gamma_n$ is a closed contour in Ω, then $\int_\Gamma f(z)\,dz = 0$

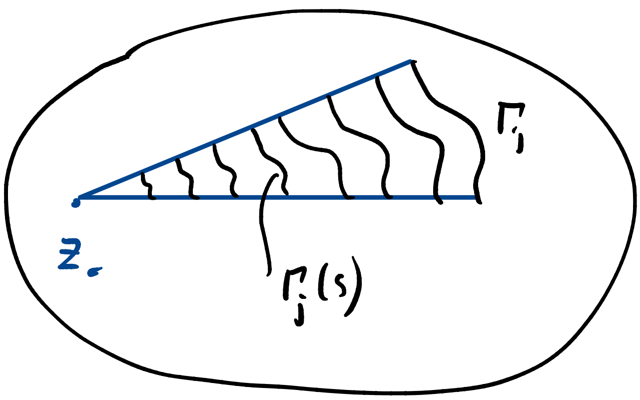
Idea of the proof: The contour is contracted into the null contour without changing the value of the integral. The integral along the null contour vanishes.



Proof: Let $\varphi_j : [0,1] \to \Gamma_j$ without loss of generality since $\psi_j : [a_j, b_j] \to \Gamma_j$ can be transformed into $\varphi_j : [0,1] \to \Gamma_j$ via $\varphi_j := \psi_j\left(a_j + t(b_j - a_j)\right)$

$$\int_\Gamma f(z)\,dz = \sum_{j=1}^{n} \int_{\Gamma_j} f(z)\,dz \quad \text{with} \quad \int_{\Gamma_j} f(z)\,dz = \int_0^1 f(\varphi_j(t))\,\varphi_j'(t)\,dt$$

The process of contraction is demonstrated explicitly for a single path segment.



$$\varphi_j(t,s) = (1-s)z_0 + s\,\varphi_j(t) = z_0 + [\varphi_j(t) - z_0]\cdot s$$

$s = 0$: z_0

$s = 1$: $\varphi_j(t)$

$$J_j(s) = \int_{\Gamma_j(s)} f(z)\,dz = \int_0^1 f(\varphi_j(t,s))\,\frac{\partial}{\partial t}\varphi_j(t,s)\,dt$$

$$\Rightarrow \quad J_j(s) = s \int_0^1 f(\varphi_j(t,s))\, \varphi_j'(t)\, dt$$

$$J_j(0) = 0, \quad J_j(1) = \int_{\Gamma_j} f(z)\, dz$$

$$\frac{d J_j(s)}{ds} = \int_0^1 f(\varphi_j(t,s))\, \varphi_j'(t)\, dt + \int_0^1 dt\, f'(\varphi_j(t,s)) \underbrace{\left(\frac{\partial}{\partial s} \varphi_j(t,s)\right)}_{\varphi_j(t) - z_0} \underbrace{\varphi_j'(t)\, s}_{\frac{\partial \varphi_j(t,s)}{\partial t}}$$

$$= \int_0^1 dt\, \frac{\partial}{\partial t} \left[f(\varphi_j(t,s)) \cdot (\varphi_j(t) - z_0) \right]$$

$$= f(\varphi_j(1,s)) \cdot [\varphi_j(1) - z_0] - f(\varphi_j(0,s)) [\varphi_j(0) - z_0]$$

In total we consider a closed contour, hence

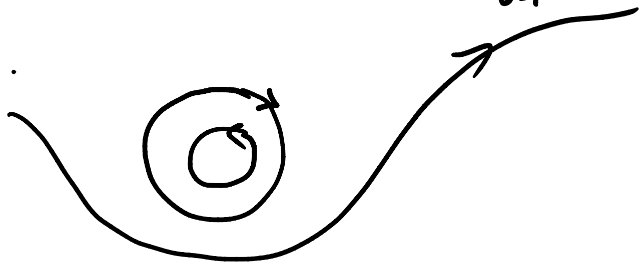
$\varphi_j(1) = \varphi_{j+1}(0)$ for $j = 1, \dots, n-1$, and $\varphi_n(1) = \varphi_1(0)$

$$\Rightarrow \frac{d}{ds} \sum_{j=1}^n J_s = \sum_{j=1}^n \frac{d J_j(s)}{ds} = 0 \Rightarrow \sum_{j=1}^n J_j(s) \text{ is constant}$$

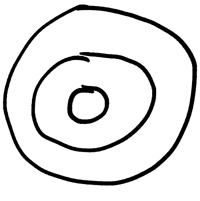
as a function of s.

$$\Rightarrow \int_\Gamma f(z)\, dz = \sum_{j=1}^n J_j(1) = \sum_{j=1}^n J_j(0) = 0$$

Let $\Gamma_1 \dots \Gamma_n$ be contours, then $\bigcup_{j=1}^n \Gamma_j$ is a contour as well.



A contour is called closed if it is the unification of closed contours, e.g.



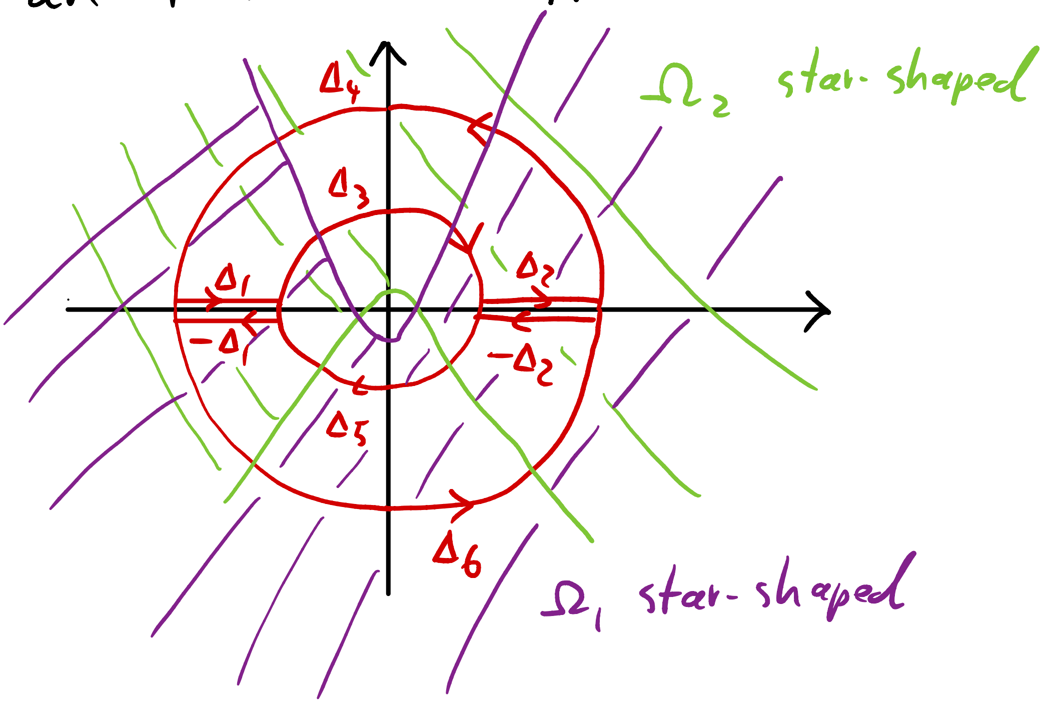
Let $\Gamma = \Gamma_1 + \dots + \Gamma_n$ be a contour, with Γ_i denoting connected subcontours (also called piecewise smooth curves; in this sense a path segment would be called a smooth curve).

Then $$\int_\Gamma f(z)dz = \sum_{j=1}^{n} \int_{\Gamma_j} f(z)dz$$

Def.: Let $\Omega \subset \mathbb{C}$ be open, Γ a closed contour in Ω. Γ is called null-homotopic with respect to Ω ($\Gamma \underset{\Omega}{\simeq} 0$), if there are closed contours $\Gamma_1, \dots, \Gamma_n$ in Ω and open and star-shaped sets $\Omega_1, \dots, \Omega_n \subset \Omega$ with $\Gamma_i \subset \Omega_i$, such that $\Gamma = \sum_i \Gamma_i$

Two contours Γ and Δ are called homotopic with regards to Ω ($\Gamma \underset{\Omega}{\simeq} \Delta$), if $\Gamma - \Delta \underset{\Omega}{\simeq} 0$

Examples: (1) $\Omega = \mathbb{C}\setminus\{0\}$. Γ is the unification of two concentric circles around the origin, which are transversed in opposite directions.



Γ can be decomposed into two closed curves,

$\Gamma = \Gamma_1 + \Gamma_2$ with $\Gamma_1 = \Delta_1 + \Delta_3 + \Delta_2 + \Delta_4 \subset \Omega_2$

$\Gamma_2 = -\Delta_1 + \Delta_6 - \Delta_2 + \Delta_5 \subset \Omega_1$

$\Rightarrow$ Γ is null-homotopic (the individual circles are not null-homotopic!) with respect to Ω.

$C_1 = \Delta_4 + \Delta_6$, $C_2 = \Delta_3 + \Delta_5$

$\Gamma = C_1 + C_2 \underset{\Omega}{\sim} 0$ $\quad C_1 \underset{\Omega}{\sim} -C_2$

all circles traversed in the same direction are homotopic with respect to Ω.

Example ② $\Omega = \mathbb{C}$ Every closed curve is null-homotopic (with respect to Ω).

Cauchy integral theorem

Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ holomorphic.

$$\Gamma \underset{\Omega}{\simeq} 0 \Rightarrow \int_\Gamma f(z)\,dz = 0$$

Proof: $\Gamma \underset{\Omega}{\simeq} 0 \Rightarrow \exists\ \Gamma_j,\ j = 1, \dots, n,\quad \Gamma = \sum_{j=1}^{n} \Gamma_j$

$\Gamma_j \subset \Omega_j$, $\Omega_j \subset \Omega$ star-shaped and open

$$\int_\Gamma f(z)\,dz = \sum_{j=1}^{N} \int_{\Gamma_j} f(z)\,dz = 0$$

Consequence: Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ holomorphic

$$\Gamma \underset{\Omega}{\simeq} \Delta \Rightarrow \int_\Gamma f(z)\,dz = \int_\Delta f(z)\,dz$$

Proof: $\Gamma \underset{\Omega}{\simeq} \Delta \Rightarrow \Gamma - \Delta \underset{\Omega}{\simeq} 0$

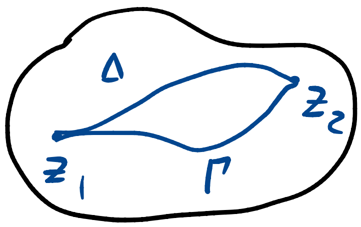
$$\Rightarrow 0 = \int_{\Gamma-\Delta} f(z)\,dz = \int_\Gamma f(z)\,dz + \int_{-\Delta} f(z)\,dz$$

$$= \int_\Gamma f(z)\,dz - \int_\Delta f(z)\,dz$$

$$\Rightarrow \int_\Gamma = \int_\Delta$$

Some applications:

① Let Γ and Δ be contours from z_1 to z_2, $\Omega \subset \mathbb{C}$ be open and star-shaped



The integral only depends on initial and final points since it has the same value for all contours. We can write for holomorphic functions in a star-shaped set $\int_{z_1}^{z_2} f(z)\,dz$

② Let $\Omega = \mathbb{C} \setminus \{0\}$, $f(z) = z^n$, $n \in \mathbb{Z}$

concentric circles around the origin

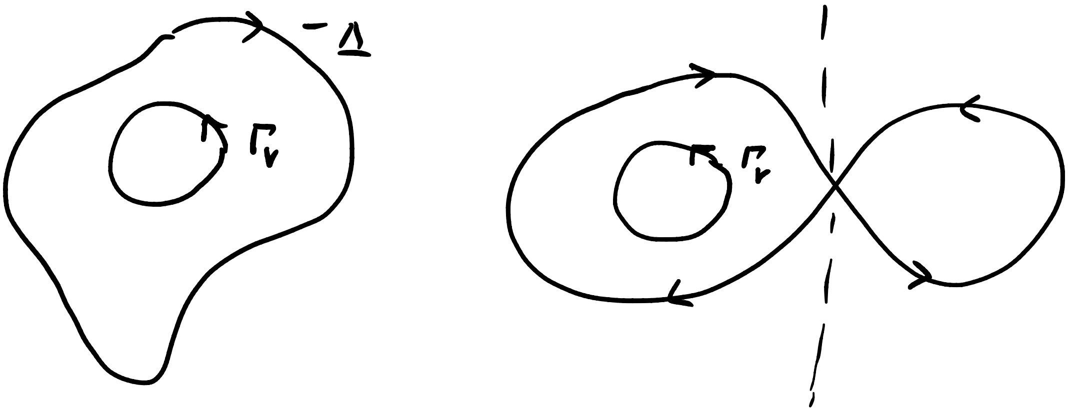
$\Gamma_r = \{ z = r e^{2\pi i t},\ 0 \leq t \leq 1 \}$, the Γ_r are not null-homotopic with respect to Ω.

$$\int_{\Gamma_r} z^n dz = \int_0^1 r^n e^{2\pi i n t}\, r 2\pi i e^{2\pi i t}\, dt$$

$$= 2\pi i r^{n+1} \int_0^1 e^{2\pi i (n+1) t}\, dt$$

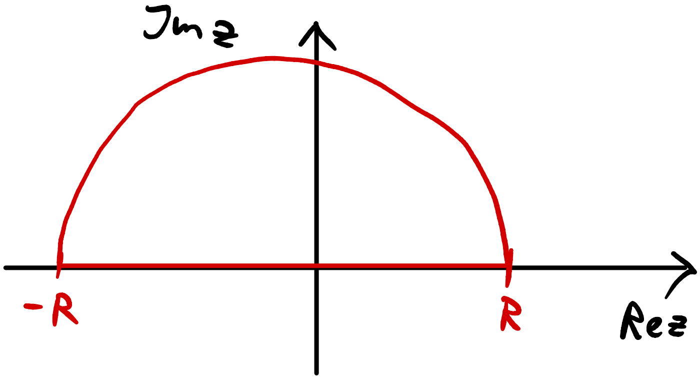
$$= \begin{cases} 0 & \text{for } n \neq -1 \\ 2\pi i & \text{for } n = -1 \end{cases}$$

Consider Δ with $\Delta \underset{\Omega}{\sim} \Gamma_r$, then $\Gamma_r - \Delta$ null-homotopic $\Rightarrow$ $\int_\Delta z^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$



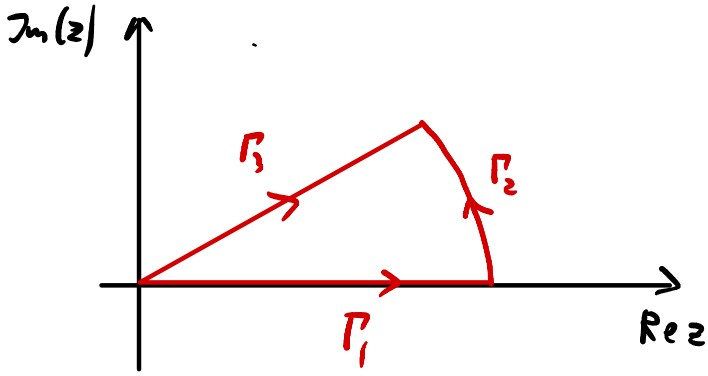
③ Evaluation of real integrals using Cauchy's integral theorem

idea:



$$\int_{-\infty}^{\infty} dx\, f(x) = \lim_{R\to\infty} \oint f(z)\,dz$$

Alternatively, consider $\Omega = \mathbb{C}$, $f(z) = e^{-z^2}$



$\Gamma = \Gamma_1 + \Gamma_2 - \Gamma_3$ closed

$\Gamma_1 = \{t : 0 \leq t \leq R\}$

$\Gamma_2 = \{Re^{it} : 0 \leq t \leq \frac{\pi}{4}\}$, $\Gamma_3 = \{t\, e^{i\frac{\pi}{4}} : 0 \leq t \leq R\}$

f is holomorphic $\Rightarrow 0 = J_1 + J_2 - J_3 = \int_{\Gamma_1} + \int_{\Gamma_2} - \int_{\Gamma_3}$

$$\Rightarrow 0 = \int_0^R e^{-t^2}dt + \int_0^{\frac{\pi}{4}} e^{-R^2 e^{2it}} R i e^{it}\,dt - \int_0^R e^{-t^2 e^{i\frac{\pi}{2}}} e^{i\frac{\pi}{4}}\,dt$$

$$\lim_{R\to\infty} J_1 = \int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

We now show that $\lim_{R\to\infty} J_2 = 0$

$$|J_2| = \left| \int_0^{\frac{\pi}{4}} e^{-R^2 e^{2it}} R i e^{it}\,dt \right|$$
$$\leq R \int_0^{\frac{\pi}{4}} \left| e^{-R^2 e^{2it}} \right| dt$$

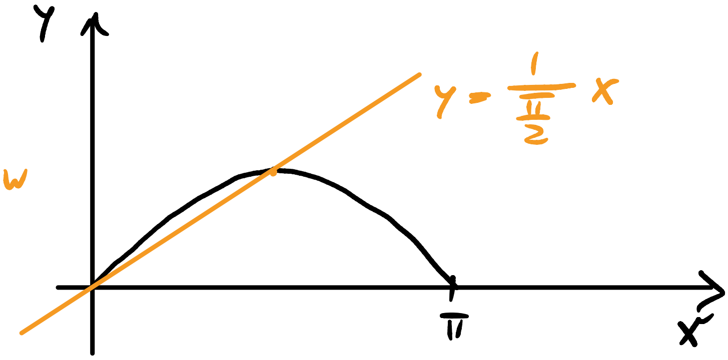
Define $\delta := R^{-\frac{3}{2}}$; for large R we have $\delta < \frac{\pi}{4}$

For $0 \leq t \leq \frac{\pi}{4} - \delta$ we find

$$\cos 2t \geq \cos\left(\frac{\pi}{2} - 2\delta\right) = \sin 2\delta \overset{(*)}{\geq} 2\delta \frac{1}{\frac{\pi}{2}}$$

Proof of $(*)$

the line is below the sine function



Use $|e^{x+iy}| = e^x$

$$\left| e^{-R^2 e^{2it}} \right| = \exp\left[\operatorname{Re}\left(-R^2 e^{2it}\right)\right] = \exp(-R^2 \underbrace{\cos 2t}_{\geq \frac{4\delta}{\pi}})$$

$$\leq \exp\left(-R^2 \frac{4\delta}{\pi}\right)$$

$$\Rightarrow |J_2| \leq R \left\{ \int_0^{\frac{4}{\pi}-\delta} |\exp(-R^2 2^{it})|\,dt + \int_{\frac{\pi}{4}-\delta}^{\frac{\pi}{4}} |\exp(-R^2 e^{2it})|\,dt \right.$$

$$\Rightarrow |J_2| \leq R \int_0^{\frac{\pi}{4}-\delta} e^{-R^2 \frac{4\delta}{\pi}} dt + R\delta$$

$$= R\left(\frac{\pi}{4}-\delta\right) \exp\left(-R^2 \frac{\pi}{4} R^{-\frac{3}{2}}\right) + R R^{-\frac{3}{2}}$$

$$\to 0 \quad \text{for } R \to \infty$$

This implies that

$$\frac{\sqrt{\pi}}{2} = \lim_{R\to\infty} J_3 = \int_0^\infty e^{i\left(\frac{\pi}{4}-t^2\right)} dt = \frac{1}{\sqrt{2}}(1+i) \int_0^\infty dt \left(\cos t^2 - i \sin t^2\right)$$

real part $\displaystyle\int_0^\infty (\cos t^2 + \sin t^2) dt = \sqrt{2} \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{2}}$

imaginary part $\displaystyle\int_0^\infty (\cos t^2 - \sin t^2) dt = 0$

$$\Rightarrow \int_0^\infty \cos t^2 dt = \frac{1}{2}\sqrt{\frac{\pi}{2}} = \int_0^\infty \sin t^2 dt$$

substitute $\boxed{s = t^2 \Rightarrow \int_0^\infty \frac{\sin s}{\sqrt{s}} ds = \sqrt{\frac{\pi}{2}}}$

Fresnel integral

2.3 Cauchy's integral formula

Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega \to \mathbb{C}$ holomorphic

For every $z_0 \in \Omega$ $\exists \delta > 0$ such that $K(z_0, \delta) \subset \Omega$

The circles $\Gamma_r = \{ z = z_0 + re^{it} : 0 \leq t \leq 2\pi \}$
for $0 < r < \delta$ are pairwise homotopic with respect to $\Omega \setminus \{z_0\}$.

Theorem: For each contour homotopic to Γ_r with respect to $\Omega \setminus \{z_0\}$ the following holds true

$$\boxed{f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} \, dz}$$

first Cauchy integral formula

Proof: $\dfrac{f(z)}{2\pi i (z - z_0)}$ is holomorphic in $\Omega \setminus \{z_0\}$

$$\int_{\Gamma_r} \frac{f(z)}{2\pi i (z - z_0)} dz = \int_{\Gamma_r} \left[\frac{f(z) - f(z_0)}{2\pi i (z - z_0)} + \frac{f(z_0)}{2\pi i (z - z_0)} \right] dz$$

① $$\int_{\Gamma_r} \frac{f(z_0)}{2\pi i (z - z_0)} dz = \frac{f(z_0)}{2\pi i} \underbrace{\int_{\Gamma_r} \frac{dz}{z - z_0}}_{2\pi i \text{ result from last lecture}} = f(z_0)$$

② $\int_{\Gamma_r} \frac{f(z)-f(z_0)}{2\pi i(z-z_0)} = 0$ Proof is analogous to that of Cauchy's integral theorem, contraction of integration path into z_0 yields due to differentiability $f'(z_0)$

Theorem: Let Γ be a contour in $\mathbb{C}$, $g : \Gamma \to \mathbb{C}$ is continuous. Then for every $n \in \mathbb{N}_0$

$$\boxed{f_n(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{g(\xi)}{(\xi - z)^{n+1}}\, d\xi}$$

is holomorphic in $\mathbb{C}\setminus\Gamma$ and $\boxed{f_n'(z) = f_{n+1}(z)}$

Proof: need to show that

$$\lim_{z\to z_0} \frac{f_n(z) - f_n(z_0)}{z - z_0} = f_{n+1}(z_0) \text{ for all } z_0 \in \mathbb{C}\setminus\Gamma$$

$$\frac{1}{(\xi - z_0)^{n+1}} - \frac{1}{(\xi - z)^{n+1}} = \frac{(\xi-z)^{n+1} - (\xi - z_0)^{n+1}}{(\xi - z_0)^{n+1}(\xi - z)^{n+1}}$$

$\ulcorner$ Now use the formula

$(a-b)(a^n + a^{n-1}b + \dots + b^n) = a^{n+1} - b^{n+1}$ $\lrcorner$

$$= \frac{(z - z_0) \sum_{j=0}^{n} (\xi - z_0)^{n-j} (\xi - z)^j}{(\xi - z_0)^{n+1}(\xi - z)^{n+1}}$$

$$\Rightarrow \frac{f_n(z) - f_n(z_0)}{z - z_0} = \frac{n!}{2\pi i} \int_\Gamma g(\xi) \frac{\sum_{j=0}^{n} (\xi - z_0)^{n-j}(\xi - z)^j}{(\xi - z_0)^{n+1}(\xi - z)^{n+1}}\, d\xi$$

converges uniformly to $\frac{n+1}{(\xi - z_0)^{n+2}}$ for $z \to z_0$

$$\xrightarrow{z \to z_0} \quad \frac{n!\,(n+1)}{2\pi i} \int_\Gamma g(\xi) \frac{1}{(\xi - z_0)^{n+2}} \, d\xi$$

Theorem: Cauchy's integral formulas

$\Omega \subset \mathbb{C}$ open, $f : \Omega \to \mathbb{C}$ holomorphic

$\Rightarrow f' : \Omega \to \mathbb{C}$ is holomorphic, and for every $z_0 \in \Omega$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \, dz \qquad n = 0, 1, 2, \dots$$

for each Γ homotopic to Γ_r. In particular, every holomorphic function is differentiable to arbitrarily high order.

Proof: first CI-formula $\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} dz = f(z_0)$

Let in the theorem above for $g := f \Rightarrow f_1 = f_0' = f'$
holomorphic

$$f_{n+1} = f_n' = \left(f^{(n)}\right)' = f^{(n+1)} \quad \text{holomorphic}$$

2.4 Antiderivatives of holomorphic functions

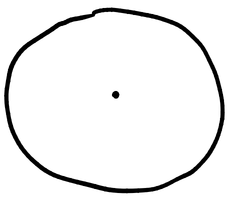
Def.: $\Omega \subset \mathbb{C}$ is called connected if for every pair of points $z_0, z_1 \in \Omega$ there exists a contour Γ from z_0 to z_1 with $\Gamma \subset \Omega$.
Ω is called simply connected if Ω is connected and every closed contour is null-homotopic.

Example ①: Ω is star-shaped $\Rightarrow$ Ω is simply connected.

Proof: $z_1, z_2 \in \Omega$, consider

$\{$path segments $z_1 \to z_0 \to z_2\} \subset \Omega$

② $\mathbb{C} \setminus \{0\}$ $\qquad$ $\mathbb{C} \setminus F$ $\qquad$ are connected, but not simply connected



Proof: Assume $\mathbb{C} \setminus \{0\}$ was simply connected

$\Rightarrow$ every closed contour in $\mathbb{C}\setminus\{0\}$ is null-homotopic

$\Rightarrow$ Γ_r is null-homotopic with respect to $\mathbb{C}\setminus\{0\}$

$$\Rightarrow \int_{\Gamma_r} z^{-1}\, dz = 0 \quad \lightning$$

Let $\Omega \subset \mathbb{C}$ be open, simply connected,
$f : \Omega \to \mathbb{C}$ continuous, and

$$\int_\Gamma f(z)\,dz = 0 \quad \text{for closed contours } \Gamma$$

Let Γ_1 and Γ_2 be contours from z_0 to z_1

$\Rightarrow \Gamma_1 - \Gamma_2$ is closed

$$\Rightarrow \int_{\Gamma_1 - \Gamma_2} f(z)\,dz = 0 \Rightarrow \int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz \equiv \int_{z_0}^{z_1} f(z)\,dz$$

Theorem: $F(z) = \int_{z_0}^{z} f(\xi)\,d\xi$ is a holomorphic function,
and $F'(z) = f(z)$ for $z \in \Omega$.

Proof: Let $z_1 \in \Omega \Rightarrow \exists_{\delta > 0}\ K(z_1, \delta) \subset \Omega$

for $z \in K(z_1, \delta)$ we have

$$F(z) = \int_{z_0}^{z_1} f(\xi)\,d\xi + \int_{z_1}^{z} f(\xi)\,d\xi$$

$$= F(z_1) + \int_{z_1}^{z} f(\xi)\,d\xi$$

$$= F(z_1) + \int_0^1 f\big((1-t)z_1 + tz\big)(z - z_1)\,dt$$

$$\Rightarrow \frac{F(z) - F(z_1)}{z - z_1} = \int_0^1 f\big((1-t)z_1 + tz\big)\,dt$$

$$\Rightarrow \quad \frac{F(z)-F(z_1)}{z-z_1} = f(z_1) + \underbrace{\int_0^1 [f((1-t)z_1 + t z_2) - f(z)]\,dt}_{\substack{\to 0 \text{ for } z \to z_1 \\ \text{due to the continuity of } f}}$$

$$z \to z_1 \qquad \left.\frac{dF}{dz}\right|_{z_1} = f(z_1)$$

Consequence: Ω simply connected, $f: \Omega \to \mathbb{C}$ holomorphic $\Rightarrow$ $F(z) = \int_{z_0}^{z} f(\xi)\,d\xi$ is holomorphic.

Morera's theorem: Let $\Omega \subset \mathbb{C}$ be open, $f: \Omega \to \mathbb{C}$ continuous

$$\boxed{f \text{ holomorphic} \iff \int_\Gamma f(z)\,dz = 0 \text{ for every } \Gamma \underset{\Omega}{\simeq} 0}$$

Proof: "$\Rightarrow$" Cauchy's integral theorem

"$\Leftarrow$" $z_0 \in \Omega \Rightarrow \exists_{\delta > 0}\; K(z_0, \delta) \subset \Omega$

$K(z_0, \delta)$ is simply connected $\Rightarrow$ every closed contour in K is null-homotopic.

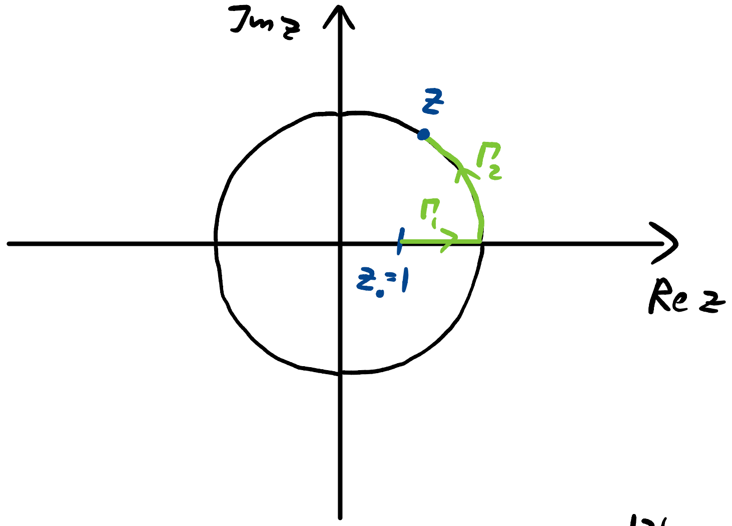
$\Rightarrow$ $F: K(z_0, \delta) \to \mathbb{C}$, $F(z) = \int_{z_0}^{z} f(\xi)\,d\xi$ is holomorphic

$\Rightarrow$ F is arbitrarily often complex differentiable

$\Rightarrow$ $F' = f$ is holomorphic.

Example: Let $\Omega = \mathbb{C} \setminus \{0\}$, $f(z) = \frac{1}{z}$ is holomorphic in Ω, but Ω is not simply connected.

Question: is there a unique antiderivative?



$$F(z) = \int_{\Gamma_1} \frac{1}{\xi} d\xi + \int_{\Gamma_2} \frac{1}{\xi} d\xi = \int_1^{|z|} \frac{1}{\xi} d\xi + \int_0^{\arg z} \frac{1}{|z| e^{it}} |z| i e^{it} dt$$

$$= \ln |z| + i \arg z = \ln z$$

Let Γ_2' be the contour which encircles the circle with radius $|z|$ once in addition to Γ_2.

$$\int_{\Gamma_1} \frac{1}{\xi} d\xi + \int_{\Gamma_2'} \frac{1}{\xi} d\xi = \ln |z| + i \int_0^{\arg z + 2\pi} dt = \ln z + 2\pi i$$

$\Rightarrow$ if Ω is not simply connected, there is no unique antiderivative in general.

§3 Series representation of holomorphic functions

3.1 Taylor series

Def.: Let $z_0 \in \mathbb{C}$, (a_j) a sequence in $\mathbb{C}$

$$f_n(z) = \sum_{j=0}^{n} a_j (z-z_0)^j \quad , \quad f_n : \mathbb{C} \to \mathbb{C} \text{ polynomial}$$

If $f_n \to f$ in $\Omega \subset \mathbb{C}$, then we write

$$f(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j$$

$\sum_{j=0}^{\infty} a_j (z-z_0)^j$ is called power series with expansion point z_0, the f_n are partial sums, the a_j are the coefficients of the series.

Theorem: the power series $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ converges uniformly in all closed discs

$\bar{D}(z_0, \rho') = \{ z : |z-z_0| \leq \rho' \}$ with

$\rho' < \rho \equiv \left[\limsup_{n\to\infty} \sqrt[n]{|a_n|} \right]^{-1}$ (and $\rho = \infty$ is allowed)

and diverges for $|z-z_0| > \rho$.

ρ is called radius of convergence, $D(z_0, \rho)$ disc of convergence. The function $f: D(z_0,\rho) \to \mathbb{C}$,

$f(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j$ is holomorphic. f can be integrated and differentiated term by term.

$$f'(z) = \sum_{j=1}^{\infty} j a_j (z-z_0)^{j-1}$$

$$F(z) = \sum_{j=0}^{\infty} \frac{a_j}{j+1} (z-z_0)^{j+1}$$

$F(z)$ is the antiderivative of f with $F(z_0)=0$.

The power series of F and f' have the same radius of convergence as f.

Proof is analogous to the one in real analysis.

Proof of divergence for $|z-z_0|>\varsigma$

Assume the series was convergent $\Rightarrow |a_n(z-z_0)|^n \xrightarrow[n\to\infty]{} 0$

$$\Rightarrow \underset{n_1\in\mathbb{N}}{\exists} \underset{n>n_1}{\forall} \quad |a_n|\cdot|z-z_0|^n < 1$$

$$\Rightarrow \sqrt[n]{|a_n|} < \frac{1}{|z-z_0|} \Rightarrow \limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq \frac{1}{|z-z_0|} < \frac{1}{\varsigma}$$

$\lightning$ to $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{\varsigma}$

Taylor's theorem: Let f be holomorphic in $K(z_0,r)$.

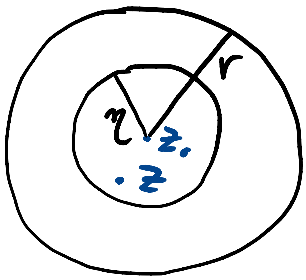
Then there exists exactly one power series $\sum_{j=0}^{\infty} a_j (z-z_0)^j$ with radius of convergence $\varsigma \geq r$, which is equal to $f(z)$ for every $|z-z_0|<r$. It holds

that $a_n = \frac{1}{n!} f^{(n)}(z_0)$. The coefficients satisfy Cauchy's coefficient bound

$$|a_n| \le \frac{1}{\eta^n} \cdot \max\{|f(z)| : |z-z_0| = \eta\} \text{ for}$$

every $\eta \in (0,r)$.

Proof: Let $|z-z_0| < r$, $\Gamma = \{z_0 + \eta e^{it}, 0 \le t \le 2\pi\}$ with $|z-z_0| < \eta < r$



$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_\Gamma f(\xi) \frac{1}{\xi - z} \, d\xi$$

For $\xi \in \Gamma$ it holds true that $\left|\frac{z-z_0}{\xi-z_0}\right| = \frac{|z-z_0|}{\eta} < 1$

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0} \frac{1}{1 - \frac{z-z_0}{\xi - z_0}} = \frac{1}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{\xi-z_0}\right)^n$$

The series $\sum_{n=0}^{\infty} \left(\frac{z-z_0}{\xi-z_0}\right)^n$ converges uniformly for $\xi \in \Gamma$

$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_\Gamma f(\xi) \frac{1}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{\xi-z_0}\right)^n d\xi$$

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \, (z-z_0)^n$$

uniform convergence

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

Coefficient bound:

$$a_n = \frac{f^{(n)}}{n!} \overset{C.I.}{=} \frac{1}{2\pi i} \int_{\Gamma_\varsigma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \quad \text{for all } \varsigma < r$$

$$|a_n| = \frac{1}{2\pi} \left| \int_{\Gamma_\varsigma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \right| \leq \frac{1}{2\pi} \underbrace{2\pi\varsigma}_{|\Gamma_\varsigma|} \frac{1}{\varsigma^{n+1}} \max\{|f(\xi)| : |\xi - z_0| = \varsigma\}$$

Identity theorem for power series

Let $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n (z-z_0)^n$ be two power series with radii of convergence r_1 and r_2

$$f: K(z_0, r_1) \to \mathbb{C}, \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$g: K(z_0, r_2) \to \mathbb{C}, \quad g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

Let $f(z_j) = g(z_j)$ for a sequence $z_j \in K(z_0, \min\{r_1, r_2\})$ with $z_j \to z_0$, $z_j \neq z_0$. Then $a_n = b_n$ for all $n \in \mathbb{N}_0$, and hence $f = g$.

Proof by induction over n:

Base case: $\quad a_0 = f(z_0) = \lim_{j\to\infty} f(z_j) = \lim_{j\to\infty} g(z_j) = g(z_0) = b_0$

Inductive hypothesis: $a_j = b_j$ for $j = 0, 1, \dots, n-1$

Need to show $a_n = b_n$

Define $f_n(z) = \sum_{j=n}^{\infty} a_j (z - z_0)^{j-n} = \dfrac{f(z) - \sum_{j=0}^{n-1} a_j (z-z_0)^j}{(z-z_0)^n}$

$$g_n(z) = \sum_{j=n}^{\infty} a_j (z-z_0)^{j-n} = \frac{g(z) - \sum_{j=0}^{n-1} b_j (z-z_0)^j}{(z-z_0)^n}$$

It holds true that $f_n(z_j) = g_n(z_j)$

$$\Rightarrow a_n = f_n(z_0) = \lim_{j\to\infty} f_n(z_j) = \lim_{j\to\infty} g_n(z_j) = g_n(z_0) = b_n$$

Example: $\displaystyle\sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j} (z-1)^j$

Radius of convergence: $\left[\overline{\lim} \sqrt[n]{\frac{1}{n}}\right]^{-1} = 1$, $\rho = 1$

$$\ln z = \sum_{n=0}^{\infty} a_n (z-1)^n \quad \text{for } |z-1| < 1$$

$a_n = \dfrac{1}{n!} \ln^{(n)}(1)$

$$\ln' z = \frac{1}{z}$$

$$\ln'' z = -\frac{1}{z^2}$$

$$\ln^{(n)}(z) = \frac{(-1)^{n+1}(n-1)!}{z^n}$$

$$\Rightarrow \quad a_n = \frac{(-1)^{n+1}(n-1)!}{n!} = \frac{(-1)^{n+1}}{n}$$

The region in which a function is holomorphic can be much larger than the radius of convergence of the power series.

Theorem: Let $\Omega \subset \mathbb{C}$ be open and connected,

$f: \Omega \to \mathbb{C}$ holomorphic,

$f \not\equiv 0$ (i.e. $\exists_{z \in \Omega}\ f(z) \neq 0$)

$\Rightarrow$ the set of zeros of f does not have a limit point in Ω.

indirect proof: assume z_0 was a limit point of zeros of f, $z_0 \in \Omega$

Ω open $\Rightarrow \underset{\varrho_0 > 0}{\exists}\ K(z_0, \varrho_0) \subset \Omega$

z_0 limit point of zeros $\Rightarrow$ there exists a sequence (z_j), $z_j \in K(z_0, \varrho_0)$, $z_j \neq z_0$ with $f(z_j) = 0$

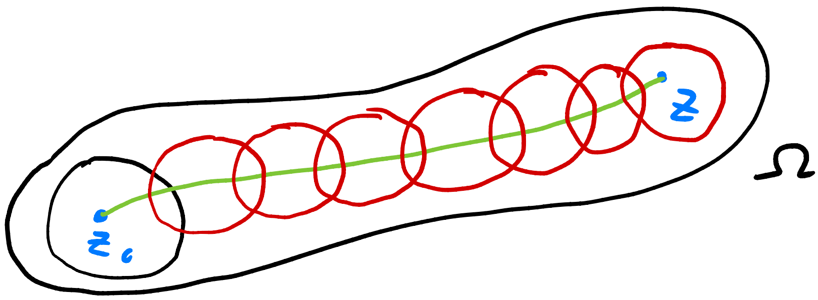
Consider the zero function $g(z) \equiv 0$,

$$g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n \,, \quad b_n = 0 \text{ for all } n$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{for } z \in K(z_0, \rho_0)$$

$$f(z_j) = 0 = g(z_j) \implies f = g \equiv 0 \text{ in } K(z_0, \rho_0)$$

Consider a $z \in \Omega$

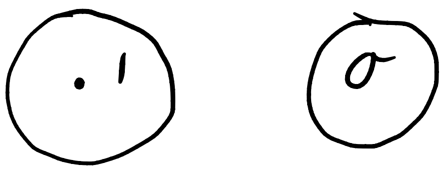


Ω is connected $\Rightarrow$ $\exists$ contour from z_0 to z

Cover the contour with overlapping discs,

for every pair of overlapping discs choose a common point in the two discs, repeat the above argument $\Rightarrow f(z) = 0 \Rightarrow f(z) \equiv 0$ ↯

Remarks: 1) Ω connected is an essential assumption, otherwise for example the following would be possible



2) A limit point of zeros in principle could lie

on the boundary of Ω $\Rightarrow$ Ω open is essential

3) $f \not\equiv c$ and holomorphic $\Rightarrow$ there is no limit point of c-point in a connected and open Ω.

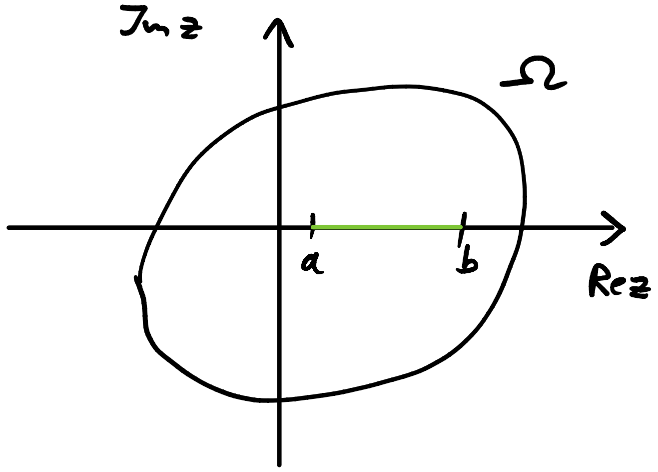
Identity theorem for holomorphic functions

Let $\Omega \subset \mathbb{C}$ be open and connected, f and g holomorphic in Ω. For a sequence (z_j) in Ω with $z_j \to z_0 \in \Omega$, $z_j \neq z_0$ we assume $f(z_j) = g(z_j)$.
Then $f(z) = g(z)$ $\forall_{z \in \Omega}$

Proof: $h(z) = f(z) - g(z)$ $\quad h(z_j) = 0$ for all j

$\Rightarrow$ $h \equiv 0$ $\Rightarrow$ $f = g$ #

Application:



$f, g : \Omega \to \mathbb{C}$ holomorphic

$f(x) = g(x)$ for $x \in (a, b)$ $\Rightarrow$ $f = g$

$\boxed{\text{A real function has at most one holomorphic extension.}}$

Definition: A function which is holomorphic in all of $\mathbb{C}$ is called entire function.

Example: $\exp : \mathbb{C} \to \mathbb{C}$ $\exp(z) = e^x(\cos y + i \sin y)$ is an entire function.

Expansion around zero: $\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\exp^{(n)}(0)}_{1} z^n$

$$\left[\overline{\lim} \sqrt[n]{\frac{1}{n!}}\right]^{-1} = \infty$$

$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$, $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

are entire functions

Theorem: Let f be an entire function. If there exist $m \in \mathbb{N}_0$, $0 < c \in \mathbb{R}$, and $0 < r_0 \in \mathbb{R}$ with

$$M(r) := \max\{|f(z)| : |z| = r\} \leq c\, r^m$$

for $r > r_0$ (i.e. f grows at most as fast as r^m)

$\Rightarrow$ f is a polynomial of degree $\leq m$.

Proof: Expand $f(z)$ around 0, $f(z) = \sum_{n=0}^{\infty} a_n z^n$

Cauchy bound $|a_n| \leq \frac{M(r)}{r^n}$

$\Rightarrow$ for $r > r_0$, $n > m$ it holds true that

$|a_n| \leq c\, r^{m-n} \xrightarrow[r \to \infty]{} 0 \Rightarrow a_n = 0$ for $n > m$

$\Rightarrow f(z) = \sum_{n=0}^{m} a_n z^n$ polynomial of degree $\leq m$

Liouville's first theorem:

A bounded entire function is constant.

Proof: f bounded $\Rightarrow |f(z)| \leq c\, r^0$

Apply the above theorem for $n = 0$.

Def.: Let f be holomorphic at z_0. z_0 is called zero of order m of f if

$$f(z) = \sum_{n=m}^{\infty} a_n (z - z_0)^n = (z - z_0)^m \sum_{n=0}^{\infty} a_{m+n} (z - z_0)^n$$

with $a_m \neq 0$

Consequence: z_0 is zero of order m of $f \Leftrightarrow$

$$f^{(n)}(z_0) = 0 \text{ for } n < m$$

"$\Rightarrow$" definition above

"$\Leftarrow$" apply Taylor's theorem

Theorem: Multiplication of power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad |z-z_0| < \rho_1$$

$$g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n \qquad |z-z_0| < \rho_2$$

$$f(z) \cdot g(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k \qquad \text{for } |z-z_0| < \min(\rho_1, \rho_2)$$

radius of convergence of $\sum_{k=0}^{\infty} c_k (z-z_0)^k$ is $\rho \geq \min(\rho_1, \rho_2)$,

and $c_k = \sum_{n=0}^{k} a_n b_{k-n}$

Proof: following Taylor's theorem we can expand

$f \cdot g$ in $K(z_0, \min(\rho_1, \rho_2))$ as

$$(f g)(z) = \sum_k c_k (z-z_0)^k$$

$$c_k = \frac{(f g)^{(k)}(z_0)}{k!} = \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} f^{(n)}(z_0)\, g^{(k-n)}(z_0)$$

$$\Rightarrow \quad c_k = \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} a_n n! \, b_{k-n} (k-n)!$$

remember $\binom{k}{n} = \frac{k(k-1)\dots(k-n+1)}{n!} = \frac{k!}{(k-n)!\,n!}$

$$\Rightarrow \quad c_k = \sum_{n=0}^{k} a_n b_{n-k} \qquad \#$$

Example for radius of convergence of $f \cdot g > \min(\varrho_1, \varrho_2)$

$f(z) = \frac{1}{1-z} \qquad (z_0 = 0, \ \varrho_1 = 1)$

$g(z) = 1-z \qquad (z_0 = 0, \ \varrho_2 = \infty)$

$f \cdot g = 1$ radius of convergence ∞

Application: f holomorphic at z_0, $f(z_0) \neq 0$

$\Rightarrow \exists_{r>0} \ f(z) \neq 0$ for $z \in K(z_0, r) \Rightarrow h = \frac{1}{f}$

holomorphic at z_0 $\Rightarrow$ can be expanded in Taylor

series $\quad h(z) = \sum_{m=0}^{\infty} b_m (z-z_0)^m \qquad z \in K(z_0, r)$

$(f \cdot h)(z) = 1 = \sum_{k=0}^{\infty} c_k (z-z_0)^k \qquad c_0 = 1, \ c_k = 0$ for $k > 0$

Let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad f(z_0) \neq 0 \Rightarrow a_0 \neq 0$

$k=0 \qquad 1 = a_0 b_0 \Rightarrow b_0 = \frac{1}{a_0}$

$k>0 \qquad 0 = \sum_{n=0}^{k} a_n b_{k-n}$

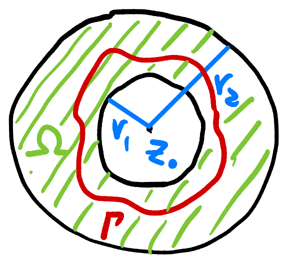
e.g. $k=1 \qquad 0 = a_0 b_1 + a_1 b_0 \Rightarrow b_1 = -\frac{a_1 b_0}{a_0}$

3.2 Laurent series

Extension of Taylor series by terms with negative indices leads us to consider Laurent series.

Theorem: Let $f : \Omega \to \mathbb{C}$ be holomorphic in $\Omega = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$ (where $r_1 = 0$, $r_2 = \infty$ are allowed). Then for $z \in \Omega$ the following holds true:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{with}$$

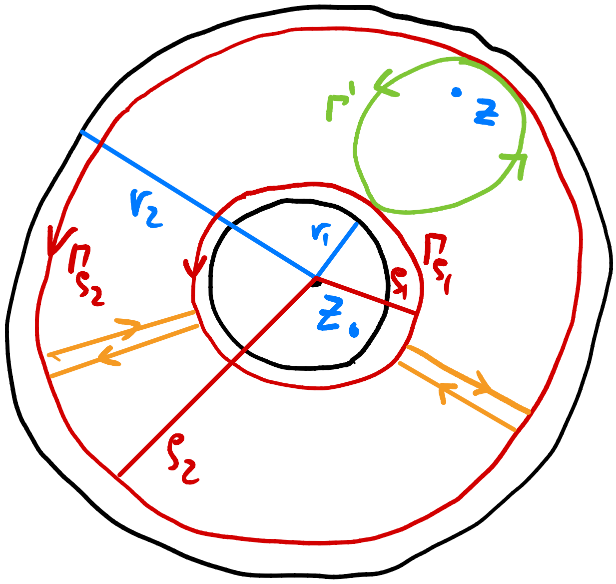


$$a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi \text{, where}$$

Γ is a circular contour $\Gamma_r = \{z = z_0 + re^{it}, 0 \le t \le 2\pi\}$ with $r_1 < r < r_2$, or a contour homotopic to Γ_r.

The series converges uniformly in closed annuli $\{z \in \mathbb{C} : \rho_1 \le |z - z_0| \le \rho_2\}$ for ρ_1 and ρ_2 with $r_1 < \rho_1 < \rho_2 < r_2$.

Proof: Choose a $z \in \Omega \Rightarrow \exists_{\, \rho_1, \rho_2,\ r_1 < \rho_1 < |z-z_0| < \rho_2 < r_2}$



$$f(z) = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\Gamma_{\rho_2}} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\Gamma_{\rho_1}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$\Gamma' \underset{\Omega \setminus \{z\}}{\sim} \Gamma_{\rho_2} - \Gamma_{\rho_1} \Rightarrow \Gamma' - \Gamma_{\rho_2} + \Gamma_{\rho_1} \underset{\Omega \setminus \{z\}}{\sim} 0$$

1st integral $\quad \zeta \in \Gamma_{\rho_2} \Rightarrow \left| \frac{z - z_0}{\zeta - z_0} \right| = \frac{|z - z_0|}{\rho_2} < 1$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \quad \text{converges uniformly}$$

$$\text{1st integral} = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \int_{\Gamma_{\rho_2}} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

2nd integral: $\quad \zeta \in \Gamma_{\rho_1} \Rightarrow \left| \frac{\zeta - z_0}{z - z_0} \right| = \frac{\rho_1}{|z - z_0|} < 1$

$$\Rightarrow \frac{1}{\xi - z} = -\frac{1}{z-z_0} \frac{1}{1 - \frac{\xi - z_0}{z - z_0}} = -\frac{1}{z-z_0} \sum_{n=0}^{\infty} \left(\frac{\xi - z_0}{z - z_0} \right)^n$$

$$= -\sum_{n=0}^{\infty} \frac{(\xi - z_0)^n}{(z - z_0)^{n+1}} \quad \text{converges uniformly}$$

$$\Rightarrow -(\text{2nd integral}) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^{-n-1} \int_{\Gamma_{S_1}} \frac{f(\xi)}{(\xi - z_0)^{-n}} \, d\xi$$

$$= \frac{1}{2\pi i} \sum_{n=-\infty}^{-1} (z - z_0)^n \int_{\Gamma_{S_1}} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n$$

$$\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{(\xi - z_0)^{n+1}} \, d\xi$$

The uniform convergence follows from the uniform convergence of the geometric series used at an intermediate step.

Def.: $\Omega = \{ z \in \mathbb{C} : 0 < |z - z_0| < r \}$, f holomorphic in Ω. z_0 is called singularity of f.

1. $a_n = 0 \quad \forall_{n<0}$ f can be holomorphically continued to z_0, z_0 is a removable singularity.

2. $a_n = 0$ for $n < -m$, $a_{-m} \neq 0$, $m \in \mathbb{N}$
   z_0 is a pole of order m of f

3. $a_n \neq 0$ for infinitely many negative indices n.
z_0 is an essential singularity.

Remarks: ① as a consequence of Liouville's theorem, it follows that a non-constant holomorphic function has at least one singularity (including singularities at infinity).

② essential singularities have many pathological features: one can show that in any small neighbourhood of an essential singularity a function $f(z)$ comes arbitrarily close to every complex value w_0

③ branch points and branch lines: similar to the logarithm, every $f(z) = z^a$ with non-integer a has a branch cut starting at zero and extending along the negative real axis. The function $f(z) = \sqrt{z^2 - 1}$ has a branch cut extending from -1 to $+1$ along the real axis.

Def.: a_{-1} is the residue of f at z_0, $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$

$$a_{-1} = \mathrm{Res}(f, z_0)$$

How to compute residues:

case 1: f has pole of order one at z_0

$$f(z) = \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$\Rightarrow \lim_{z \to z_0} f(z-z_0)(z-z_0) = a_{-1}$$

case 2: f has a pole of order $m>1$. Then

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \left\{ \frac{d^{m-1}}{dz^{m-1}} \left[f(z)(z-z_0)^m \right] \right\}$$

Proof: $$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \dots + \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$\lim_{z \to z_0} \frac{1}{(m-1)!} \left(\frac{d}{dz}\right)^{m-1} \Big(a_{-m} + a_{-m+1}(z-z_0)^1 + \dots + a_{-m+k}(z-z_0)^k + \dots + a_{-1}(z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} \Big)$$

$$= \lim_{z \to z_0} \frac{1}{(m-1)!} \Big[(m-1)!\, a_{-1} + \underbrace{\sum_{n=0}^{\infty} (n+m)(n+m-1) \cdot \ldots \cdot (n+2) a_n (z-z_0)^{n+1}}_{\to 0} \Big]$$

$$= a_{-1}$$

Examples: ① $f(z) = \frac{1}{z}$, $z_0 = 0$ is pole of first order, $a_{-1} = \operatorname{Res}(f, 0) = 1$

$f(z) = \frac{1}{z^2}$, $z_0 = 0$ is pole of second order, $\operatorname{Res}(f, 0) = 0$

② $\Omega = \mathbb{C} \setminus \{0\} \quad f(z) = e^{\frac{1}{z}}, \quad z \in \Omega$

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n$$

$a_n = \frac{1}{(-n)!}$ for $-\infty < n \leq 0$

$\mathrm{Res}(f, 0) = 1$

3.3 The residue theorem

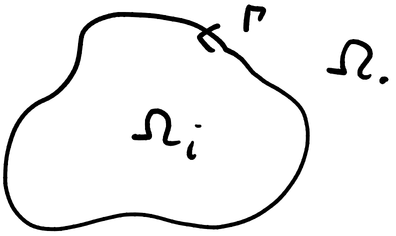
Def.: A contour is called simple if it does not intersect itself, meaning it has no double points (with the possible exception of the beginning and end points for closed contours.
A simple closed curve (contour) is called Jordan curve.

Jordan curve theorem:

Let Γ be a Jordan curve. Then there exist two open connected sets Ω_i, Ω_o such that

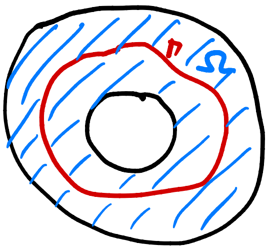
$\mathbb{C} = \Gamma \cup \Omega_i \cup \Omega_o$, $\quad \Gamma \cap \Omega_i = \emptyset$, $\Gamma \cap \Omega_o = \emptyset$

$\Omega_i \cap \Omega_o = \emptyset$



In addition, Ω_i is simply connected.

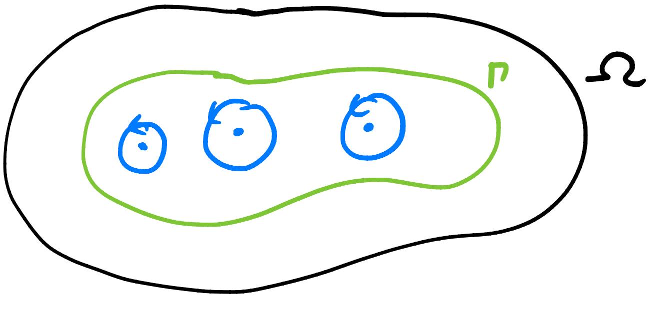
Def.: A Jordan curve is called positively oriented if Ω_i consists of exactly the points $z_0 \in \mathbb{C}\setminus\Gamma$ for which Γ is homotopic to $\Gamma_{z_0} = \{z = z_0 + \varrho e^{it},\ 0 \leq t \leq 2\pi\}$ with respect to $\mathbb{C}\setminus\{z_0\}$.



is not allowed

Let $\Omega \subset \mathbb{C}$, Γ is a Jordan curve and $\Gamma \cup \Omega_i \subset \Omega$

Let $z_1, \dots, z_n \in \Omega_i \Rightarrow \exists$ circles Γ_j around z_j which lie in Ω_i and do not intersect each other pairwise.



It holds true that $\Gamma \underset{\Omega\setminus\{z_i\}}{\sim} \Gamma_1 + \Gamma_2 + \dots \Gamma_n$ if

Γ is positively oriented.

Residue theorem: Let Γ be a positively oriented Jordan curve with $z_1, \dots, z_n \in \Omega_i$.
$\Omega \subset \mathbb{C}$ open with $\Gamma \cup \Omega_i \subset \Omega$.
If $f: \Omega \setminus \{z_1, \dots, z_n\}$ is holomorphic, then

$$\boxed{\int_\Gamma f(z)dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j)}$$

Proof: $f: \Omega \setminus \{z_1, \dots, z_n\} \to \mathbb{C}$ holomorphic

$$\xRightarrow{\text{C-I theorem}} \int_{\Gamma - \sum_j \Gamma_j} f(z) dz = 0$$

$$\Rightarrow \int_\Gamma f(z)dz = \sum_j \int_{\Gamma_j} f(z) dz$$

Laurent-expansion of f around z_j

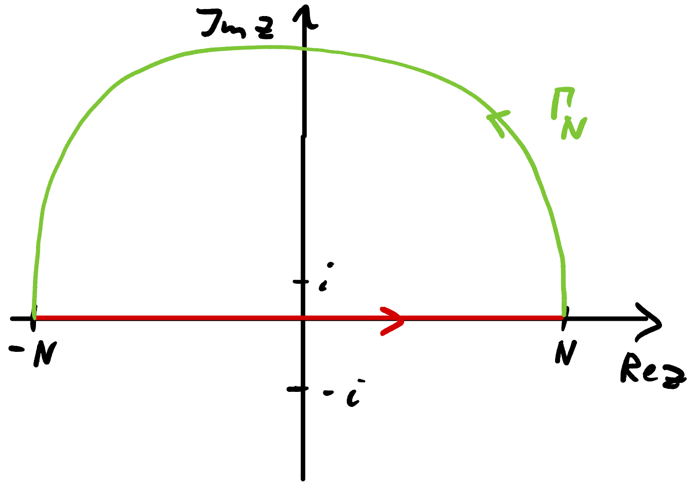
$$a_{-1} = \mathrm{Res}(f, z_j) = \frac{1}{2\pi i} \int_{\Gamma_j} \frac{f(z)}{(z-z_j)^0} dz = \frac{1}{2\pi i} \int_{\Gamma_j} f(z)dz$$

$$\Rightarrow \int_\Gamma f(z)dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(f, z_j) \quad \#$$

The residue theorem is important for computing integrals along the real axis.

Examples: ① $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi$

Proof:



singularities at $\pm i$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \lim_{N\to\infty} \int_{-N}^{N} \frac{1}{x^2+1}\,dx$$

$$= \lim_{N\to\infty}\left[\oint_{[-N,N]+\Gamma_N} \frac{1}{z^2+1}\,dz - \int_{\Gamma_N} \frac{1}{z^2+1}\,dz \right]$$

$$\operatorname{Res}\left[\frac{1}{(z+i)(z-i)},\, i\right] = \frac{1}{2i}$$

$$\Rightarrow \lim_{N\to\infty}[\ldots] = \underbrace{2\pi i\,\frac{1}{2i}}_{\pi} - \underbrace{\lim_{N\to\infty}\int_{\Gamma_N}\frac{dz}{z^2+1}}_{\to 0 \text{ is to demonstrate}}$$

$$\left|\int_{\Gamma_N}\frac{dz}{z^2+1}\right| \underset{\uparrow\ \varphi(t)=Ne^{it}}{=} \left|\int_0^{\pi} dt\,\frac{1}{N^2e^{2it}+1}\,Ni\,e^{it}\right|$$

$$\leq \int_0^{\pi} \frac{N}{|N^2e^{2it}+1|}\,dt$$

Use now that $|N^2 e^{2it}| - 1 \leqslant |N^2 e^{2it} + 1|$

$$\leqslant \int_0^{\pi} \frac{N}{N^2-1} dt = \frac{N\pi}{N^2-1} \xrightarrow[N\to\infty]{} 0$$

Assumptions for applicability:

i) The number of enclosed singularities has to stay finite for $N \to \infty$

ii) $|f(z)| \lesssim |z|^{-(1+\varepsilon)}$ for $|z| \to \infty$ with an $\varepsilon > 0$ has to be satisfied.

Example ② Computation of integrals

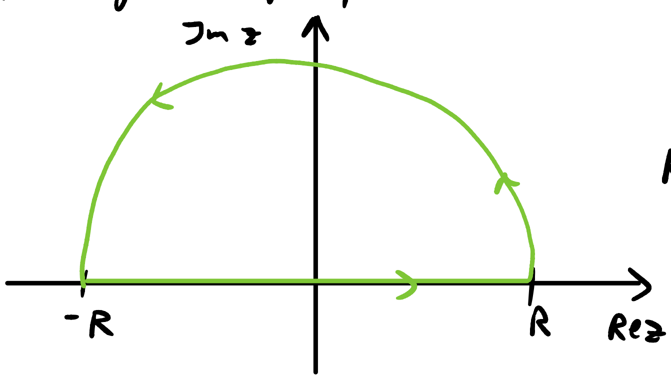
$$J = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx \qquad \text{(Fourier transform)}$$

for $\alpha \in \mathbb{R}^+$

We assume that i) $f(z)$ is holomorphic in the upper half plane with the exception of a finite number of poles, ii) $\lim\limits_{|z|\to\infty} f(z) = 0$, $0 \leq \arg z \leq \pi$

This assumption is less restrictive than in ①

We again employ a semi-circular contour



$R \to \infty$

The residue theorem is applied in the same way as before, but we have to work harder to show that the integral J_R over the semicircle vanishes.

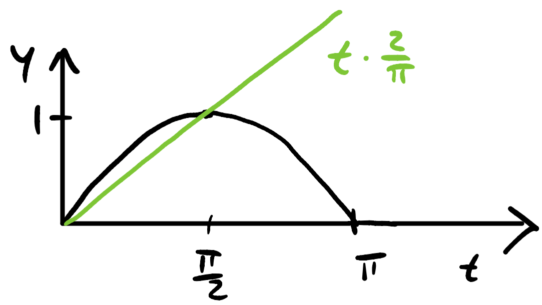
$$J_R = \int_0^{\pi} f(Re^{it})\, e^{iaR\cos t - aR\sin t}\, iRe^{it}\, dt$$

Let R be large enough such that $|f(z)| = |f(Re^{it})| < \varepsilon$

$$\Rightarrow \quad |J_R| \leq \varepsilon R \int_0^{\pi} e^{-aR\sin t}\, dt$$

$$= 2\varepsilon R \int_0^{\pi/2} e^{-aR\sin t}\, dt$$

In the range $[0, \frac{\pi}{2}]$ we have $\frac{2}{\pi} t \leq \sin t$



$$\Rightarrow \quad |J_R| \leq 2\varepsilon R \int_0^{\pi/2} e^{-aR2t/\pi}\, dt$$

$$= 2\varepsilon R \, \frac{1 - e^{-aR}}{aR2/\pi}$$

$$\Rightarrow \quad \lim_{R\to\infty} |J_R| \leq \frac{\pi}{a} \varepsilon$$

As $\varepsilon \to 0$ as $R \to \infty$, we find $\lim_{R\to\infty} |J_R| = 0$

This result is sometimes called Jordan's lemma

$$\Rightarrow \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = 2\pi i \sum_{z_i,\ \operatorname{Im} z_i > 0} \operatorname{Res}(f, z_i)$$

§ 4 Fourier transformation and the δ-function

Def.: Let $f: \mathbb{R} \longrightarrow \mathbb{R}$, then

$$\tilde{f}(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iqx} \, dx$$

is the Fourier transform of the function f.

Def.: Let $f, g: \mathbb{R} \to \mathbb{R}$. Then

$$F(x) := \int_{-\infty}^{\infty} f(y) g(x-y)$$ is the convolution of f with g.

<u>Convolution Theorem</u>: Let $f(x)$ and $g(x)$ be real valued functions and $\tilde{f}(q)$, $\tilde{g}(q)$ their Fourier transforms. Then $\sqrt{2\pi}\, \tilde{f}(q) \tilde{g}(q)$ is the Fourier transform of the convolution $F(x)$ of f with g.

Proof: $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)\, e^{-iqx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; f(y) g(x-y) e^{-iqx}$$

Rewrite the integrand as $f(y) e^{-iqy} \; g(x-y)\, e^{-iq(x-y)}$

Substitute $s := x - y$, $ds = dx$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, f(y) e^{-iqy}}_{\tilde{f}(q)} \; \underbrace{\int_{-\infty}^{\infty} ds\; g(s)\, e^{-iqs}}_{\sqrt{2\pi}\; \tilde{g}(q)}$$

δ- function

Review: Kroneck - δ

ⓐ $\delta_{j,0} = \begin{Bmatrix} 0 & j \neq 0 \\ 1 & j = 0 \end{Bmatrix} \equiv \delta(j)$

Considering $\delta_{j,0}$ is not a restriction due to

$$\delta_{j,k} = \delta_{j-k,0} = \begin{cases} 0 & j-k \neq 0, \text{ i.e. } j \neq k \\ 1 & j-k = 0, \text{ i.e. } j = k \end{cases}$$

Alternative definition

ⓑ $\sum_{j \in \mathbb{Z}} f_j \delta(j) = f_0$ ⓐ $\Leftrightarrow$ ⓑ

for arbitrary sequences f_j

Proof: ⓐ $\Rightarrow$ ⓑ trivial

$\Leftarrow$ consider sequences $(f_j) := (\ldots 000 \ldots 010 \ldots 0$

position k, $k \in \mathbb{Z}$

$\Rightarrow \delta(k) = 0$ for $k \neq 0$

$\delta(k=0) = 1$

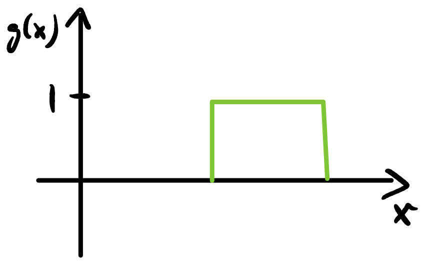
Definition of δ- function $\delta(x)$

Replace $j \longrightarrow x \in \mathbb{R}$ in ⓑ

$\int_{-\infty}^{\infty} f(x)\delta(x) = f(x=0)$ for arbitrary (well behaved) functions $f(x)$.

Specifically: $f(x)\equiv 1 \Rightarrow \int\limits_{-\infty}^{\infty}\delta(x)\,dx = 1$

Consider $g(x) = \begin{cases} 1: & x \in I, \text{ where } I \text{ is an arbitrary interval with } 0 \in I \\ 0: & x \notin I \end{cases}$



$$\int\limits_{-\infty}^{\infty} g(x)\,\delta(x)\,dx = g(x=0) = 0$$

corresponds to $\int\limits_{I,\,0\notin I} dx\,\delta(x) = 0$

Generalized function or distribution:

Interpret $\int\limits_{-\infty}^{\infty} f(x)\delta(x)\,dx$ as short version of

$\lim\limits_{n\to\infty} \int\limits_{-\infty}^{\infty} f(x)\delta_n(x)\,dx$, where the $\delta_n(x)$

are regular functions.

1st representation: $\delta_n(x) = \begin{cases} \frac{n}{2} & : \ |x| \leq \frac{1}{n} \\ 0 & : \ |x| > \frac{1}{n} \end{cases}$

It holds that $\delta_n(x) \xrightarrow[n\to\infty]{} 0 \quad \forall_{x\neq 0}$

$$\delta_n(0) \xrightarrow[n\to\infty]{} \infty$$

$$\int\limits_{-\infty}^{\infty} \delta_n(x)\,dx = \int\limits_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2}\,dx = \frac{n}{2}\left(\frac{1}{n}+\frac{1}{n}\right) = 1$$

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\delta_n(x)\,dx$$

$$= \lim_{n\to\infty} \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{n}{2} f(x)\,dx$$

$$= \lim_{n\to\infty} \frac{n}{2} \int_{-\frac{1}{n}}^{\frac{1}{n}} dx \left[f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \dots \right]$$

$$= \lim_{n\to\infty} \left[\frac{n}{2} f(0)\frac{2}{n} + \underbrace{f'(0)\frac{n}{2}\left[\frac{x^2}{2}\right]_{-\frac{1}{n}}^{\frac{1}{n}}}_{\to 0} + \underbrace{\dots}_{0} \right]$$

$$= f(0) \quad \#$$

Claim: $\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = \int_{-\infty} f(x)\delta(-x)\,dx$

Proof: $\delta_n(x) = \delta_n(-x) \ \forall_n$

Claim: $\int_{-\infty}^{\infty} f(x)\delta(x-y)\,dx = f(y) = \int_{-\infty} f(x)\delta(y-x)\,dx$

Proof: substitute $s := x-y$, $\quad x = s+y$

$ds = dx$, $\quad s$ from $-\infty$ to ∞

$$\int_{-\infty}^{\infty} f(x)\delta(x-y)\,dx = \int_{-\infty}^{\infty} f(s+y)\delta(s)\,ds = f(s+y)\big|_{s=0}$$

$$= f(y) \quad \#$$

substitution is defined for each n, in the end we consider the limit $n\to\infty$

2nd representation: Gaussian

$$\delta_n(x) = \sqrt{\frac{n}{\pi}}\, e^{-nx^2}, \quad \delta_n(0) = \sqrt{\frac{n}{\pi}} \xrightarrow[n\to\infty]{} \infty$$

$$\delta_n(x) \xrightarrow[n\to\infty]{} 0$$
$x \neq 0$

claim: the above is a representation of the δ-function

$$\int_{-\infty}^{\infty} f(x)\,\delta_n(x) = \int_{-\infty}^{\infty} f(0)\,\delta_n(x)\,dx + \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \frac{f^{(k)}(0)}{k!} x^k e^{-nx^2} dx$$

substitute $s := \sqrt{n}\, x \qquad dx = \frac{1}{\sqrt{n}} ds \;, \quad x = \frac{s}{\sqrt{n}}$

$$\int_{-\infty}^{\infty} \frac{f^{(k)}(0)}{k!} x^k e^{-nx^2} dx$$

$$= \underbrace{\frac{1}{\sqrt{n}} \frac{1}{(n)^{k/2}}}_{\to 0} \underbrace{\int_{-\infty}^{\infty} s^k e^{-s^2} ds}_{\text{finite}}$$

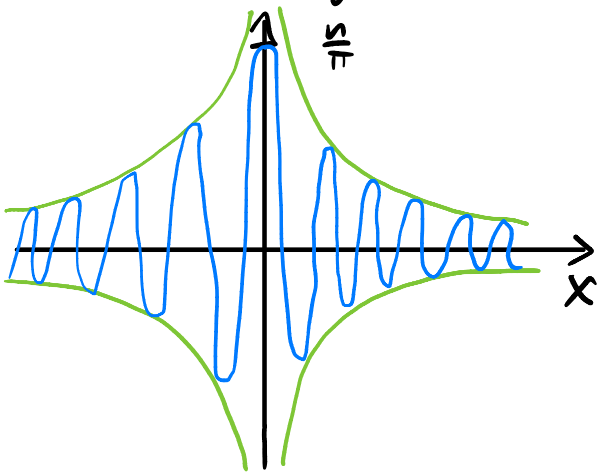
$$\Rightarrow \lim_{n\to\infty} \int f(x)\,\delta_n(x)\,dx = f(0)$$

$\delta_n(x) \xrightarrow[n\to\infty]{} \delta(x)$ is to be understood as

$$\int_{-\infty}^{\infty} f(x)\,\delta_n(x)\,dx \longrightarrow \int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$$

Only the statement about integrals is valid.

3rd representation:



$$\delta_n(x) = \frac{1}{\pi} \frac{\sin nx}{x}$$

for small x $\quad \sin(nx) = nx + \frac{(nx)^3}{3!}$

$$\delta_n(0) = \frac{n}{\pi} \xrightarrow[n\to\infty]{} \infty$$

For $n \to \infty$ the functions $\delta_n(x)$ oscillate faster and faster, but $\delta_n(x) \not\to 0$

$$\int_{-\infty}^{\infty} \frac{1}{\pi} \frac{\sin nx}{(nx)} d(xn) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{\sin s}{s} ds = 1$$

$$\delta(x) = \lim_{n\to\infty} \frac{1}{\pi} \frac{\sin nx}{x}$$ (to be understood as statement about integrals)

$$= \lim_{n\to\infty} \frac{1}{\pi} \frac{1}{2ix}\left(e^{inx} - e^{-inx}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-n}^{n} dq\, e^{iqx}$$

$$\Rightarrow \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iqx} dx$$

Theorem: Let $f: \mathbb{R} \to \mathbb{R}$ and let $\tilde{f}(q)$ be the Fourier transform of f. Then

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(q)\, e^{iqx} dq$$

Proof: $$\tilde{f}(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(q)\, e^{-iqx} dq$$

$$\Rightarrow \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(q)\, e^{iqx}\, dq = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dy\, f(y)\, e^{iq(x-y)}$$

$$= \int_{-\infty}^{\infty} dy\, f(y) \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iq(x-y)}\, dq}_{\delta(x-y)} = f(x) \quad \#$$

Remarks: ① the FT of $\delta(x)$ is $\tilde{\delta}(q) = \frac{1}{\sqrt{2\pi}}$

② to make the proof rigorous, we would need to insert a regularizer $\lim_{\eta \to 0} e^{-\eta q^2}$ before interchanging the integrals, and would then a representation $\delta_\eta(x-y)$ for the δ-function, and in the last step $\lim_{\eta \to 0} \int_{-\infty}^{\infty} dy\, f(y)\, \delta_\eta(x-y) = f(x)$.

Parsival's theorem: $\displaystyle\int_{-\infty}^{\infty} |\hat{f}(q)|^2 dq = \int_{-\infty}^{\infty} |f(x)|^2 dx$

Proof: $\displaystyle \hat{f}(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iqx}\, dx$

$$\Rightarrow \quad \int_{-\infty}^{\infty} |\hat{f}(q)|^2 dq = \int_{-\infty}^{\infty} dq \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iqx}\, dx \right) \times$$

$$\times \overline{\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\, e^{-iqy}\, dy \right)}$$

$$= \int_{-\infty}^{\infty} dx\, f(x) \int_{-\infty}^{\infty} dy\, \overline{f(y)} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} dq\, e^{iq(y-x)}}_{\delta(x-y)}$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{\overline{f(x)}}$$

$$= \int_{-\infty}^{\infty} dx\, |f(x)|^2 \qquad \#$$

Theorem on the Fourier transform of the derivative

Let $f: \mathbb{R} \to \mathbb{R}$ and $\tilde{f}(q)$ the FT of $f(x)$

In addition, let $f_n(q) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iqx} \frac{d^n f(x)}{dx^n} dx$

If $f(x), f'(x), \ldots, f^{(n-1)}(x) \xrightarrow[|x| \to \infty]{} 0$, then

$$f_n(q) = (iq)^n \cdot \tilde{f}(q)$$

Proof: $n=1$

$$f_1(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iqx} \frac{df(x)}{dx} dx$$

$$= \left[\frac{1}{\sqrt{2\pi}} e^{-iqx} f(x) \right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-iq)\, e^{-iqx} f(x)\, dx$$

$$\Rightarrow f_1(x) = (iq)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iqx} f(x)dx$$

$$= (iq)\,\tilde{f}(q)$$

Consequence: $$\frac{d^n f(x)}{dx^n} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(iq)^n \tilde{f}(q)\, e^{iqx} dx \quad (*)$$

Apply this relation formally to the δ-function:

$$\boxed{\frac{d^n \delta(x)}{dx^n} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(iq)^n e^{iqx}\, dq}$$

to be understood inside an integral

What does the integral relation imply explicitly?

$$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x)dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}(iq)^n\int_{-\infty}^{\infty} f(x)\, e^{iqx} dx\, dq$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(iq)^n \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i(-q)x} dx}_{\tilde{f}(-q)}\, dq$$

Substitute: $p := -q$, $dp = -dq$, $p = +\infty, \ldots -\infty$

$$= \frac{1}{\sqrt{2\pi}}\underbrace{(-1)\int_{\infty}^{-\infty} dp}_{\int_{-\infty}^{\infty} dp}(-ip)^n\, \tilde{f}(p)$$

$$= \frac{(-1)^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \, (ip)^n \tilde{f}(q)$$

comparison with $(*)$ $\Rightarrow$ $$\int_{-\infty}^{\infty} f(x) \frac{d^n \delta(x)}{dx^n} dx = (-1)^n \frac{d^n f}{dx^n}\Big|_{x=0}$$

Special case: $$\int_{-\infty}^{\infty} f(x) \delta'(x) dx = -f'(0)$$

Translate the center of the δ-function

$$\int_{-\infty}^{\infty} f(x) \delta'(x-a) dx$$

subst. $y := x - a$, $dy = dx$

$$= \int_{-\infty}^{\infty} dy \, f(y+a) \delta'(y) = -f'(a)$$

Alternative derivation:

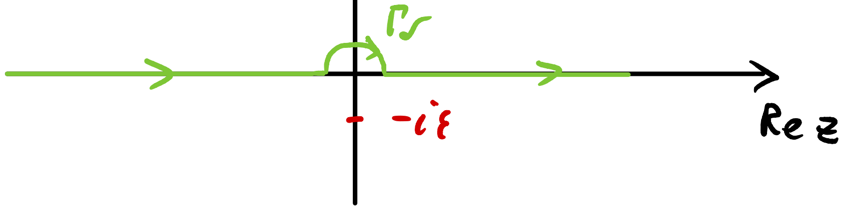
$$\int_{-\infty}^{\infty} f(x) \delta'(x-a) dx = \left[f(x) \delta(x-a) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x) \delta(x-a) = -f'(a)$$

Application: $$\boxed{\lim_{\varepsilon \to 0} (x \pm i\varepsilon)^{-1} = P \frac{1}{x} \mp i\pi \delta(x)}$$

Dirac relation

integral statement: $$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{f(x)}{x \pm i\varepsilon} dx = P \int_{-\infty}^{\infty} \frac{f(x)}{x} dx \mp i\pi f(0)$$

Proof:



denominator $x + i\varepsilon$ hat pole for $x = -i\varepsilon$

when integrating along Γ_{δ}, we can execute $\lim\limits_{\varepsilon \to 0}$

$$\int_{-\infty}^{\infty} \frac{f(x)}{x + i\varepsilon}\, dx = \lim_{\delta \to 0} \left[\int_{-\infty}^{-\delta} \frac{f(x)}{x}\, dx + \int_{\delta}^{\infty} \frac{f(x)}{x}\, dx \right] +$$

$$+ \lim_{\delta \to 0} \int_{\Gamma_\delta} \frac{f(z)}{z}\, dz$$

The first limit defines the principal value

Integral along Γ_δ: $z(t) = x_0 + \delta e^{-it} \qquad -\pi \leq t \leq 0$

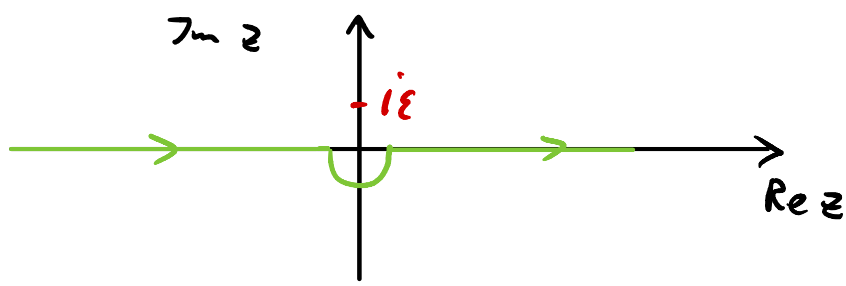
$$z'(t) = -i\delta e^{-it}$$

$$\Rightarrow \int_{\Gamma_\delta} \frac{f(z)}{z}\, dz = \int_{-\pi}^{0} \frac{f(\delta e^{-it})}{\cancel{\delta e^{-it}}} (-i) \cancel{\delta e^{-it}}\, dt$$

$$\xrightarrow[\delta \to 0]{} -i \int_{-\pi}^{0} f(0)\, dt = -i\pi f(0)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{f(x)}{x + i\varepsilon}\, dx = \underline{P} \int_{-\infty}^{\infty} \frac{f(x)}{x}\, dx - i\pi f(0) \quad \#$$

Denominator $x - i\pi$



Criteria for Fourier-Transformability

① If $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$, then the FT of f is a regular function (i.e. no distribution)

② If $\int |f(x)|^2 dx < \infty$, then the FT of f is a regular function (i.e. no distribution)

Proof in homework problem: $\frac{1}{\sqrt{x}} \longleftrightarrow \hat{f} = \frac{1}{\sqrt{q}}$

Theorem: f has a Fourier transform $\Longleftrightarrow$ f is a generalized function.

§ 5 The method of steepest descent

Often it is important to determine the asymptotic behavior of a function for large arguments, e.g. the Stirling formula for the Gamma function. The method discussed in this chapter is based on the use of complex variables and complex integrals.

Example: the Stirling formula for $s!$

$$s! = \Gamma(s+1) = \int_0^\infty t^s e^{-t}\, dt$$

substitute $t = s\tau$, $dt = s\,d\tau$

$$\overset{\downarrow}{=} s^{s+1} \int_0^\infty e^{-s(\tau - \ln\tau)}\, d\tau = s^{s+1} \int_0^\infty e^{-sf(\tau)}\, d\tau$$

$f(\tau) = \tau - \ln\tau \longrightarrow +\infty$ for $\tau \to 0$ or $\tau \to \infty$

$\Rightarrow$ for large s the integrand vanishes at the boundaries of the domain of integration.

The function $f(\tau)$ has a minimum for

$$0 \overset{!}{=} \frac{d}{d\tau}(\tau - \ln\tau) = 1 - \frac{1}{\tau} \Rightarrow \tau_0 = 1$$

$$\frac{d^2}{d\tau^2}\left(\tau - \ln\tau\right) = +\frac{1}{\tau^2} > 0 \qquad f''(1) = 1$$

$\Rightarrow$ for large s the integral is dominated by a neighborhood of $\tau_0 = 1$. Perform a Taylor expansion

$$f(\tau) = f(\tau_0) + \frac{1}{2} f''(\tau_0)(\tau-\tau_0)^2 + \frac{1}{3!} f'''(\tau_0)(\tau-\tau_0)^3 + \dots$$

$$\Rightarrow s! \approx s^{s+1} e^{-sf(\tau_0)} \int_0^\infty e^{-s\left[\frac{1}{2}f''(\tau_0)(\tau-\tau_0)^2 + \frac{1}{3!}f'''(\tau_0)(\tau-\tau_0)^3 + \dots\right]} d\tau$$

substitute $x = \sqrt{s}(\tau-\tau_0) \Rightarrow \tau = \frac{x}{\sqrt{s}} + \tau_0$, $\tau = 0$ yields $x = -\sqrt{s}$

$$\downarrow$$

$$= s^{s+1} e^{-s} \frac{1}{\sqrt{s}} \int_{-\sqrt{s}}^{\infty} e^{-\frac{1}{2}f''(\tau_0)x^2 + \frac{1}{\sqrt{s}}\frac{1}{3!}f'''(\tau_0)x^3 + \dots} dx$$

becomes small for $s \to \infty$

$$\underset{s\to\infty}{=} \frac{1}{\sqrt{s}} s^{s+1} e^{-s} \int_{-\infty}^{\infty} e^{-\frac{1}{2}f''(\tau_0)x^2} dx \quad ; \quad f''(\tau_0) = 1$$

$$= \sqrt{\frac{2\pi}{s}}\, s^{s+1} e^{-s} = \sqrt{2\pi s}\, s^s e^{-s}$$

$$\boxed{\text{Stirling formula} \quad s! \approx \sqrt{2\pi s}\, s^s e^{-s}}$$

More generally, we consider the asymptotic behavior of a function

$$J(s) = \int_{\mathcal{C}} g(z)\, e^{s f(z)}\, dz ,$$

which is defined as an integral over a contour $\mathcal{C}$ in the complex plane. For now we assume $s > 0$ to be real. We further assume that $f(z) \longrightarrow -\infty$ at the end points of the contour, or that the contour is closed. In addition we assume that $g(z)$ is dominated by the exponential function in the region of interest.

The modulus of the integrand is determined by $\mathrm{Re}\, f(z)$ (for $s \gg 1$). For $s \longrightarrow \infty$, the integral is dominated by a small region around a positive maximum of $f(z)$. Let $f(z) = u(x,y) + i v(x,y)$

$$J(s) = \int_{\mathcal{C}} g(z)\, e^{s u(x,y)}\, e^{i s v(x,y)}\, dz$$

In addition we demand that $v(x,y) \approx v(x_0, y_0) = v$ in a neighborhood of the maximum at $z_0 = x_0 + i y_0$

$$\Rightarrow J(s) \simeq e^{i s v_0} \int_{\mathcal{C}} g(z)\, e^{s u(x,y)}\, dz$$

$u(x_0, y_0)$ is a maximum for

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_0} = \left.\frac{\partial u}{\partial y}\right|_{y=y_0} = 0,$$ and due to the

Cauchy-Riemann equations $$\left.\frac{df(z)}{dz}\right|_{z=z_0} = 0$$

The maximum of $u(x_0, y_0)$ however can only be a maximum along a given contour: since $u(x,y)$ satisfies the Laplace equation, we have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

If $\frac{\partial^2 u}{\partial x^2} < 0$ (condition for maximum in x-direction),

then $\frac{\partial^2 u}{\partial y^2} > 0 \Rightarrow$ minimum in y-direction.

Holomorphic functions f without singularities can only have saddle points for this reason.

The contour we are looking for has to satisfy two conditions: i) along the contour $u(x,y)$ has to have a maximum. ii) the contour has to pass through the saddle point in such a way that the imaginary part satisfies $v(x,y) \approx v_0$.

For holomorphic functions $f(z)$ the countours with u = const. and v = const. are orthogonal to each other. For this reason, the curve $v \equiv v_0$ is tangential to the gradient of u, ∇u. For this reason, the curve $v(x,y) \equiv v_0$ determines the steepest descent from the saddle point.

The function f can be expanded into a Taylor series

$$f(z) = f(z_0) + \frac{1}{2}(z-z_0)^2 f''(z_0) + \dots$$

Along the desired contour, $\frac{1}{2}(z-z_0)^2 f''(z_0)$ is real and negative. For $f''(z_0) \neq 0$ we then have

$$f(z) - f(z_0) = \frac{1}{2}(z-z_0)^2 f''(z_0) = -\frac{1}{2s}t^2$$

In polar coordinates we find $z - z_0 = \delta e^{i\alpha}$

$$t^2 = -s f''(z_0)\delta^2 e^{2i\alpha}$$

Since t is real, we have $t = \pm \delta |s f''(z_0)|^{\frac{1}{2}}$, and

$$J(s) \approx g(z_0)\, e^{s f(z_0)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} \frac{dz}{dt}\, dt$$

Due to $(z-z_0)^2 = \delta^2 e^{2i\alpha}$, and due to

$\frac{1}{2} \delta^2 e^{2i\alpha} f''(z_0) < 0$ we find $\boxed{\alpha = \frac{\pi}{2} - \frac{1}{2} \arg f''(z_0)}$

$$f''(z_0) = - e^{-2i\alpha} |f''(z_0)|,$$

and with

$$z - z_0 = \frac{1}{|s f''(z_0)|^{\frac{1}{2}}} e^{i\alpha} t$$

$$\frac{dz}{dt} = |s f'(z_0)|^{-\frac{1}{2}} e^{i\alpha}$$

$$\Rightarrow \boxed{J(s) \approx \frac{g(z_0)\, e^{s f(z_0)}\, e^{i\alpha}}{|s f''(z_0)|^{\frac{1}{2}}}}$$