Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow
F. Puster

Mathematical Methods of Modern Physics - Problem Set 8

Summer Semester 2024

Due: The problem set will be discussed in the seminars on 06.06. and 07.06.

Internet: The problem sets can be downloaded from
https://home.uni-leipzig.de/stp/Mathematical_methods_2_ss24.html

26. Cauchy integral formula *1+2 Points*

Let C be the positively oriented unit circle. Calculate $\oint_C f(z)dz$ for

$$a)\quad f(z) = \frac{1}{z^2+4z} \qquad b)\quad f(z) = \frac{2}{2z^2+3z-2}\ .$$

27. Partial fractions and Cauchy integral formula *3 Points*

Make a partial fraction decomposition and evaluate

$$\oint_C \frac{f(z)}{z(2z+1)^2}dz,$$

for C being the positively oriented unit circle.

28. Integrating along a circle *2+2 Points*

Use the Cauchy integral formula to show that if f is holomorphic in an open set $\Omega \subset \mathbb{C}$ that includes the circle $|z-z_0| = r$,

a) then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} dt f(z_0 + re^{it})\ .$$

b) And more generally it holds that

$$f^{(n)}(z_0) = \frac{n!}{2\pi r^n}\int_0^{2\pi} dt f(z_0 + re^{it})e^{-int}\ .$$

29. Maximum modulus principle *3+1+1 Points*

a) Let f be a function that is holomorphic within a closed contour C and continuous on C. If $|f(z)| \leq M$ on C, show that

$$|f(z)| \leq M,$$

for all z within C.
Hint: Assume that there is a local maximum z_0, such that $\forall z$ within $C : |f(z_0)| \geq |f(z)|$ and $|f(z_0)| > M$. And use the result of 28 a) to find a contradiction.

b) Let f be a function that is holomorphic within a closed contour C and continuous on C. If $f(z) \neq 0$ within the contour and $|f(z)| \geq M > 0$ on C, show that

$$|f(z)| \geq M,$$

for all z within C.
Hint: Consider $w(z) = \frac{1}{f(z)}$.

c) If $f(z) = 0$ within the contour C, the statement in 29 b) does not hold anymore, i.e., it is possible to have $|f(z)| = 0$ at one or more points in the interior with $|f(z)| > 0$ over the entire bounding contour. Show this by citing a specific example of a holomorphic function that behaves this way.