Active Matter Physics

Prepare the solutions for the seminar on 2024.5.15

Exercise Sheet 3

3.1

The position $x(t)$ of a particle executing a uniform random walk is the solution of the stochastic differential equation

$$\mathrm{d}x(t) = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \qquad x(0) = x_0, \tag{1}$$

where μ and σ are constants. Find the probability density $f(t, x(t))$ of $x(t)$ at time $t > 0$. You may use the following hints:

1. Set up the corresponding Fokker–Planck equation for $f(t, x(t))$. What is the respective initial condition?
2. Apply the method of Fourier transformation as well as its properties to solve this partial differential equation.

3.2

Consider a biased random walk on a discrete lattice with $L+1$ sites and for all $1 \le i \le L-1$ the transition probabilities:

$$\pi(i \to j) = \begin{cases} p & \text{if } j = i+1 \\ q & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases} . \tag{2}$$

For the boundaries, it holds that

$$\pi(0 \to j) = \begin{cases} p & \text{if } j = 1 \\ q & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \pi(L \to j) = \begin{cases} p & \text{if } j = L \\ q & \text{if } j = L-1 \\ 0 & \text{otherwise} \end{cases} , \tag{3}$$

i.e., the random walker stays where it is if it tries to move below 0 or above L (reflecting boundary conditions).

- Calculate the stationary probability $P(0)$ using the detailed balance condition.
- Consider the explicit case of $L = 2$ and $p = 1 - q = 1/3$. Write down the transition matrix and the transition graph of this process (cf., for example, Markov chains). Calculate the stationary probability distribution $P = (P(0), P(1), P(2))$ for this case. Is the process reversible (i.e., is the detailed balance condition fulfilled)?
- In another setting a random walker obtains the following transition matrix:

  $$\Pi = (\pi(i \to j))_{i,j \in \{0,1,2\}} = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/3 & 0 & 2/3 \\ 2/3 & 1/3 & 0 \end{pmatrix} . \tag{4}$$

  Write down its transition graph. Calculate the stationary probability distribution via the ansatz $P \cdot \Pi \overset{!}{=} P$. Is this random walker in detailed balance?

3.3

We consider a micro-particle that is able to move on a discrete lattice with L lattice sites such that the site $L+1$ corresponds to the site 1 (periodic boundary conditions, see Figure 1). The particle is able to move to an ascending lattice site in clockwise direction at a rate p, whereas it moves to a lattice site in anti-clockwise direction at a rate q with $p+q=1$ (Figure 1). Such a rate refers to a probability per time entailing in a time-continuous frame, that the term "rate p" refers to the probability $p\Delta t$ of a hopping in clockwise direction within a time interval Δt. The probability to find such an asymmetric random walker at site $\sigma \in \{1, 2, ..., L\}$ (given by the Boltzmann distribution for equilibrium systems) in its stationary state is given by $P(\sigma) = 1/L$. Calculate for which values p and q the asymmetric random walker is a non-equilibrium system in the stationary state.

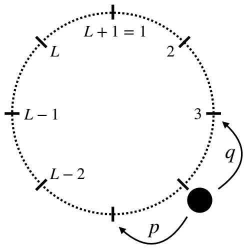

Figure 1: Asymmetric random walk with periodic boundary conditions.