Climate Dynamics (Summer Semester 2019)
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Today's Lecture (Lecture 3): Atmosphere

Reference
Peixoto and Oort, Sec. 3.1, 3.2, 3.4, 3.5; skip discussion of oceans until one week later

Description of the state of the atmosphere

- Constituents – dry air, reactive gases, absorbing gases, aerosols, water vapor, liquid water, ice
- Thermodynamic state – pressure, temperature, mixing ratios
- Dynamic state – velocity field
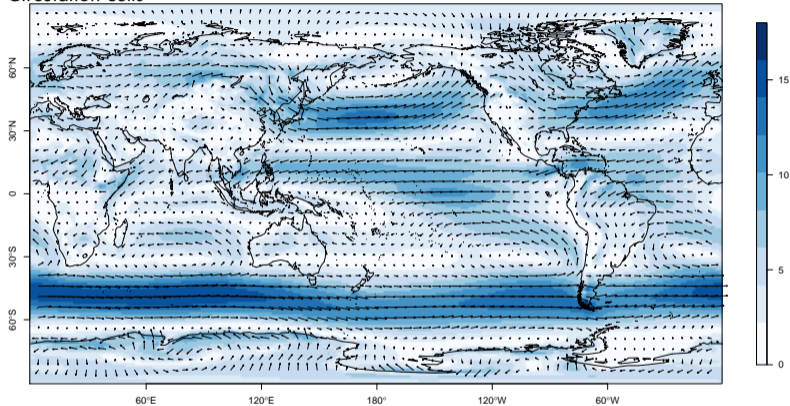- Circulation

Some features of the atmospheric circulation
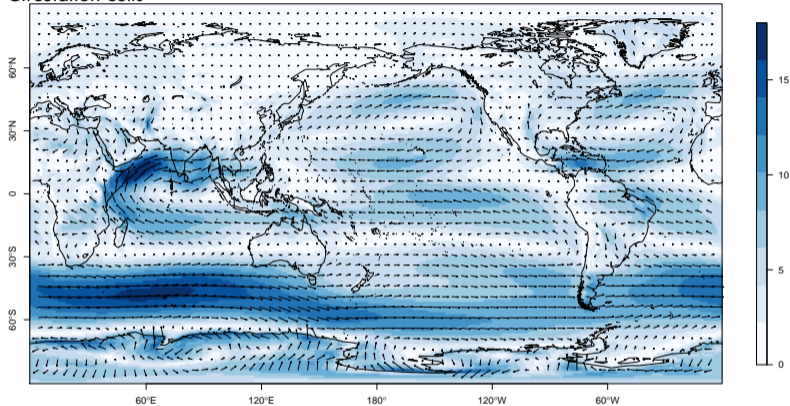
▶ Circulation cells

U.S. National Oceanic and Atmospheric Administration

▶ Eddies
▶ Boundary layer
▶ Clouds

Some features of the atmospheric circulation

- ▶ Circulation cells

- ▶ Eddies
- ▶ Boundary layer
- ▶ Clouds

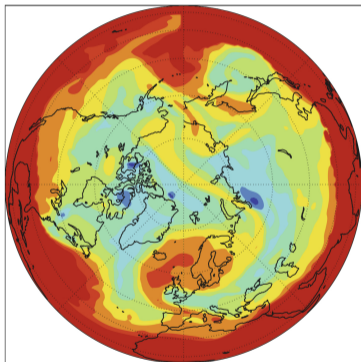# Some features of the atmospheric circulation

- ► Circulation cells

- ► Eddies
- ► Boundary layer
- ► Clouds

Some features of the atmospheric circulation

- Circulation cells
- Eddies

θ_2: Feb 14, 1994

- Boundary layer
- Clouds

Figure: Barnes et al., 2012

Some features of the atmospheric circulation

- Circulation cells
- Eddies
- Boundary layer

Figure: Hartmann, 1994
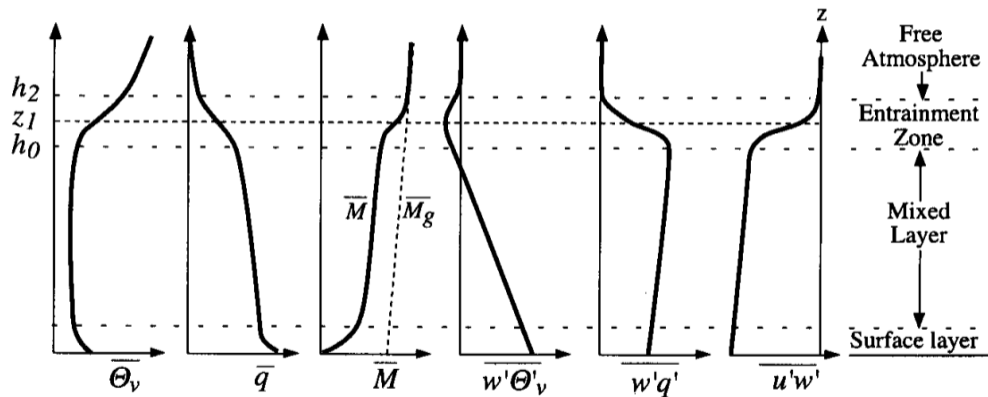
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Some features of the atmospheric circulation
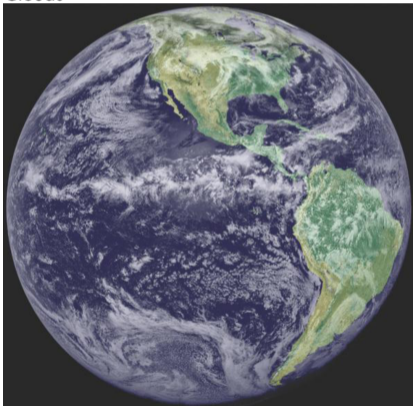
- Circulation cells
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Figure: Bellon, 2013

Equations of state

To good approximation, atmospheric gases are ideal

Dry air $\quad p_d = \rho_d R_d T$

Water vapor $\quad e = \rho_v R_v T$ (e is water vapor pressure)

Combination of dry air and water vapor $\quad p = \rho R_d T_v$ with *virtual temperature* T_v, derivation below:

The ratio of gas constants is reciprocal to the ratio of molecular weights of water vapor and dry air,

$$\frac{R_d}{R_v} = \frac{m_w}{m_d} = 0.622 \tag{2.12}$$

The total pressure and density are additive:

$$p = p_d + e \tag{2.13}$$

$$\rho = \rho_d + \rho_v = \frac{p - e}{R_d T} + \frac{e}{R_v T} = \frac{p - e}{R_d T} + 0.622 \frac{e}{R_d T} = \frac{p}{R_d T} \left(1 - 0.378 \frac{e}{p} \right) \tag{2.14}$$

(2.14) has the form of an ideal-gas equation of state with

$$T_v = \frac{T}{1 - 0.378 \frac{e}{p}} \approx T \left(1 + 0.378 \frac{e}{p} \right) = T (1 + 0.61 q), \quad \text{where} \tag{2.15}$$

$$q = \frac{M_v}{M} = \frac{\rho_v}{\rho} = 0.622 \frac{e}{(p - 0.378 e)} \approx 0.622 \frac{e}{p} \quad \text{(the water vapor \textit{mixing ratio})} \tag{2.16}$$

Atmosphere: fundamental equations

Physical principles

Conservation of mass applies to each constituent (dry air, water) individually $\rightarrow$ continuity equation

Conservation of momentum $\rightarrow$ equation of motion

Conservation of energy $\rightarrow$ thermodynamic heat equation

Conservation of angular momentum $\rightarrow$ vorticity equation

Coordinate systems

All the coordinate systems we use in this course are non-inertial (co-moving with the Earth)

Rectangular *local* coordinates: x eastward, y northward, z upward

Pressure coordinates: replace z (upward-pointing) with p (downward-pointing); any other vertical coordinate that is locally monotonic also works; $\delta p = -\rho g\, \delta z$ (hydrostatic equilibrium)

Spherical coordinates: λ (longitude), ϕ (latitude), p; $\delta x = R_E \cos\phi\, \delta\lambda$, $\delta y = R_E\, \delta\phi$

Associated differential operators

Total derivative d/dt

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{c} \cdot \nabla, \quad \text{with} \quad \vec{c} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Gradient ∇

$$\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}, \quad [\nabla] = \mathsf{m}^{-1}$$

"Horizontal" ∇_p

$$\nabla_p = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ 0 \end{pmatrix}_p, \quad (p \text{ means on isobars})$$

Continuity equation

Conservation of mass (of a dry air parcel):

$$\frac{1}{\delta m}\frac{d(\delta m)}{dt} = \frac{1}{\rho\,\delta V}\frac{d(\rho\,\delta V)}{dt} = \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{\delta V}\frac{d(\delta V)}{dt} = 0 \tag{2.17}$$

Recognizing the relative rate of expansion of the air parcel as the divergence of the wind field,

$$-\frac{1}{\rho}\frac{d\rho}{dt} = \nabla \cdot \vec{c} \tag{2.18}$$

In pressure coordinates $\delta m = \delta x\,\delta y\,\rho\,\delta z = \delta x\,\delta y\,\delta p/g$ (in hydrostatic equilibrium), and thus the wind field is non-divergent (meaning we can treat the atmosphere as incompressible):

$$\frac{1}{\delta m}\frac{d(\delta m)}{dt} = \frac{1}{\delta x\,\delta y}\frac{d(\delta x\,\delta y)}{dt} + \frac{1}{\delta p}\frac{d(\delta p)}{dt} = \nabla_p \cdot \vec{v} + \frac{\partial \omega}{\partial p} = 0, \quad \text{with} \quad \vec{v} = \left(\begin{array}{c} u \\ v \end{array}\right), \quad \omega = \frac{dp}{dt} \tag{2.19}$$

In spherical coordinates:

$$\frac{1}{R_E \cos\phi}\frac{\partial u}{\partial \lambda} + \frac{1}{R_E \cos\phi}\frac{\partial(v\cos\phi)}{\partial \phi} + \frac{\partial \omega}{\partial p} = 0 \tag{2.20}$$

Equation of motion

In an inertial frame (designated by $_A$), the forces acting on an air parcel are the pressure gradient force $-\frac{1}{\rho}\nabla p$, gravitational acceleration $-\nabla\Phi_N$ and friction $\vec{F}$:

$$\frac{d_A\vec{c}_A}{dt} = -\frac{1}{\rho}\nabla p - \nabla\Phi_N + \vec{F} \tag{2.21}$$

Consider a point with position vector $\vec{r}_r$ in a rotating frame; the transformation to position in the inertial frame includes motion of the point due to rotation of the frame:

$$\frac{d_A\vec{r}_A}{dt} = \frac{d\vec{r}_r}{dt} + \vec{\Omega}\times\vec{r}_r \quad\text{or}\quad \frac{d_A}{dt} = \frac{d}{dt} + \vec{\Omega}\times \tag{2.22}$$

or

$$\vec{c}_A = \vec{c} + \vec{\Omega}\times\vec{r}_r. \tag{2.23}$$

Applying the transformation (2.22) to $\vec{c}_A$ gives the acceleration in the inertial reference frame:

$$\frac{d_A\vec{c}_A}{dt} = \frac{d\vec{c}_A}{dt} + \vec{\Omega}\times\vec{c}_A = \frac{d}{dt}(\vec{c} + \vec{\Omega}\times\vec{r}_r) + \vec{\Omega}\times(\vec{c} + \vec{\Omega}\times\vec{r}_r) = \frac{d\vec{c}}{dt} + 2\vec{\Omega}\times\vec{c} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r}_r) \tag{2.24}$$

By a vector identity, the last term can be written as

$$\vec{\Omega}\times(\vec{\Omega}\times\vec{r}_r) = \frac{1}{2}\nabla(\vec{\Omega}\times\vec{r}_r)^2 = \frac{1}{2}\nabla(\Omega R_E\cos\phi)^2 \tag{2.25}$$

In the rotating reference frame, the equation of motion then becomes

$$\frac{d\vec{c}}{dt} = -2\vec{\Omega} \times \vec{c} - \frac{1}{\rho}\nabla p - \nabla\Phi + \vec{F},$$ (2.26)

where the term $\nabla\Phi$ has subsumed the centrifugal term $\vec{\Omega} \times (\vec{\Omega} \times \vec{r}_r)$ by (2.25) into the "apparent geopotential"

$$\Phi = \Phi_N - \frac{1}{2}\Omega^2 R_E^2 \cos^2\phi.$$ (2.27)

In component form, the equation of motion is

$$\frac{du}{dt} - \frac{uv\tan\phi}{R_E} + \frac{uw}{R_E} = -\frac{1}{\rho}\frac{1}{R_E\cos\phi}\frac{\partial p}{\partial\lambda} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_{r\lambda}$$ (2.28)

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{R_E} + \frac{vw}{R_E} = -\frac{1}{\rho}\frac{1}{R_E}\frac{\partial p}{\partial\phi} - 2\Omega u\sin\phi + F_{r\phi}$$ (2.29)

$$\frac{dw}{dt} - \frac{u^2 + v^2}{R_E} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + 2\Omega u\cos\phi + F_{rz}$$ (2.30)

Simplifying approximations

Hydrostatic approximation

Neglecting all but the highest-order terms in the vertical equation of motion yields a diagnostic relationship between the vertical pressure gradient and density:

$$\frac{\partial p}{\partial z} = -\rho g \tag{2.31}$$

The equation of motion in pressure coordinates then simplifies with the following substitutions (note that there is no vertical acceleration under the hydrostatic approximation):

$$\omega = \frac{dp}{dt} = \frac{\partial p}{\partial t} + \vec{v} \cdot \nabla p + w \frac{\partial p}{\partial z} \approx w \frac{\partial p}{\partial z} \approx -\rho g w \tag{2.32}$$

$$\left.\frac{\partial p}{\partial \lambda}\right|_z = \left.\frac{\partial p}{\partial z}\right|_\lambda \left.\frac{\partial z}{\partial \lambda}\right|_p \approx -\rho \left.\frac{\partial \Phi}{\partial \lambda}\right|_p \tag{2.33}$$

Geostrophic approximation

Proceeding similarly in the horizontal yields a diagnostic relationship between the horizontal wind and pressure gradients (in the height coordinate system) or geopotential gradients (in the pressure coordinate system):

$$f v_g = \frac{1}{\rho} \frac{1}{R_E \cos\phi} \frac{\partial p}{\partial \lambda} = \frac{1}{R_E \cos\phi} \frac{\partial \Phi}{\partial \lambda} \quad \text{and} \quad f u_g = -\frac{1}{\rho} \frac{1}{R_E} \frac{\partial p}{\partial \phi} = -\frac{1}{R_E} \frac{\partial \Phi}{\partial \phi} \tag{2.34}$$

The quality of the geostrophic approximation depends on $\text{Ro} = U/fL \ll 1$; not the case in the tropics ($f \ll 1$) or for small-scale phenomena ($fL \ll U$). $f = 2\Omega \sin\phi$ is the *Coriolis parameter*.

Thermodynamic energy equation

Combined first and second laws of thermodynamics for a reversible process (all variables are *state functions*):

$$dU = T\,dS - p\,dV \tag{2.35}$$

Since U is a state function, so is the sum (or difference) of U and other state functions:

$$H = U + pV \qquad\qquad \text{enthalpy} \tag{2.36}$$
$$dH = T\,dS + V\,dp \tag{2.37}$$
$$G = H - TS \qquad\qquad \text{Gibbs function} \tag{2.38}$$
$$dG = -S\,dT + V\,dp \tag{2.39}$$

Which differential is most convenient depends on the process under consideration:

$$\text{isentropic}\quad S = \text{const} \Rightarrow T\,dS = 0$$
$$\text{isobaric}\quad p = \text{const} \Rightarrow V\,dp = 0$$
$$\text{isothermal}\quad T = \text{const} \Rightarrow S\,dT = 0$$

For our (atmospheric) purposes, the extensive variables are replaced by their intensive counterparts. For example, (2.35) becomes

$$du = T\,ds - p\,d\alpha \tag{2.40}$$

Heat capacities

For an ideal gas, U is a function only of T. The same is true of $H = U + pV$, since pV can be related to T by the equation of state. At constant volume, any heat added to the system leads directly to an increase in internal energy, since the second term in (2.35) vanishes; we therefore define the *heat capacity at constant volume*

$$C_V = \left.\frac{dU}{dT}\right|_V \quad \text{or} \quad c_V = \left.\frac{du}{dT}\right|_V \tag{2.41}$$

At constant pressure, any heat added to the system leads directly to an increase in enthalpy, since the second term in (2.37) vanishes; we therefore define the *heat capacity at constant pressure*

$$C_p = \left.\frac{dH}{dT}\right|_p \quad \text{or} \quad c_p = \left.\frac{dh}{dT}\right|_p \tag{2.42}$$

(Note that the notation $\left.\frac{d}{dT}\right|_V$ and $\left.\frac{d}{dT}\right|_p$ is not necessary since the differentials are total differentials.)

Differentiating the definition of (specific) enthalpy with respect to temperature yields

$$c_p = \frac{dh}{dT} = \frac{d}{dT}(u + p\alpha) = \frac{d}{dT}(u + R_d T) = c_V + R_d \tag{2.43}$$

Furthermore, for dry air at atmospheric temperatures, $u \approx \frac{5}{2}R_d T$ (see Section 2.5), so

$$c_V = \frac{5}{2}R_d \quad \text{and} \quad c_V = \frac{7}{2}R_d \tag{2.44}$$

Thermodynamic energy equation for atmospheric dynamics

The most convenient expression of the thermodynamic energy balance for atmospheric dynamics is derived as follows. Recall the differentials form of the first and second laws of thermodynamics in enthalpy form:

$$dh = T\,ds + \alpha\,dp \tag{2.45}$$

Taking the time derivative and substituting from (2.42), we find

$$c_p \frac{dT}{dt} = T\frac{ds}{dt} + \alpha\frac{dp}{dt} = Q + \alpha\omega, \tag{2.46}$$

where Q is the diabatic heating rate by clouds and radiation, and $\alpha\omega$ is the heating due to adiabatic compression.

Potential temperature

For an adiabatic process, divide (2.46) by T and substitute for α/T from the equation of state:

$$c_p \frac{d\ln T}{dt} = R_d \frac{d\ln p}{dt} \tag{2.47}$$

which implies conservation of the *potential temperature* θ,

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa}, \quad \kappa = \frac{R_d}{c_p} \approx \frac{2}{7} \tag{2.48}$$

From (2.46) and (2.48) it can be shown that θ is related to entropy (up to additive constants) by

$$s = c_p \ln\theta \tag{2.49}$$