Semantics

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Counting

Properties of functions

injective each lhs is paired with a *unique* rhs (no two lhs' have the same rhs)

$$\text{if } x \neq y \text{ then } f(x) \neq f(y)$$

surjective each element of codomain is paired with some lhs

for all $y \in B$, there is some $x \in A$ such that $f(x) = y$

bijective injective and surjective

Given $f : A \to B$

If f is an *injection*
then B must be *at least as* large as A

If f is a *surjection*
then A must be *at least as* large as B

If f is a *bijection*
then A and B must be the same size

Counting

works great in finite case

$$[n] := \{1, \ldots, n\}$$

- $[0] := \emptyset$
- $[1] := \{1\}$
- $[2] := \{1, 2\}$

cardinality
$|A| = n$ iff there is a bijection $f : [n] \to A$

Infinity

Counting is weird amongst infinities

- all numbers vs odd numbers?
- numbers vs pairs of numbers (vs triples of numbers)?
- numbers vs sets of numbers?

Theorem (Cantor's lemma)
$A < 2^A$

Proving Cantor's lemma

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1. assume we had a surjection $A \to 2^A$
   - that is, assume A is at least as big as 2^A
2. show this leads to a contradiction

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- define $X := \{a : a \notin f(a)\}$

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- define $X := \{a : a \notin f(a)\}$
- question: what maps to X?

Open problems

Open problems

1. unifying constructions
2. *and*, *or* and *not* across categories

Unifying constructions

subjects

$$\begin{aligned} [\![NP\ VP]\!] &= 1 \text{ iff } [\![NP]\!] \in [\![VP]\!] \\ &= f_S([\![NP]\!], [\![VP]\!]) \end{aligned}$$

objects

$$\begin{aligned} [\![V\ NP]\!] &= \{x : \langle x, [\![NP]\!]\rangle \in [\![V]\!]\} \\ &= f_{VP}([\![V]\!], [\![NP]\!]) \end{aligned}$$

what do f_S and f_{VP} have to do with one another?

And, Or, Not

sentences

$$[\![S_1 \text{ and } S_2]\!] = [\![S_1]\!] \mathbin{\&} [\![S_2]\!]$$

VPs

$$[\![VP_1 \text{ and } VP_2]\!] = [\![VP_1]\!] \cap [\![VP_2]\!]$$

NPs

$$[\![NP_1 \text{ and } NP_2]\!] = ???$$

transitive Vs

$$[\![V_1 \text{ and } V_2]\!] = ???$$

what does and mean?

Unifying constructions

Unifying constructions

subjects

$$f_S(x, A) = 1 \text{ iff } x \in A$$

objects

$$f_{VP}(x, R) = \{y : \langle y, x \rangle \in R\}$$

what do they have in common?

strategy
change our perspective

Change of perspective

If you can't say something in two ways
you can't say it at all

- started with sets ($\wp(E),\ \wp(E \times E)$)
- change to functions ($[E \to 2],\ [E \to [E \to 2]]$)

Notation

$[A \to B]$
the set of all functions with

- *domain A*
- *codomain B*

$\mathbf{0} = \emptyset$
the empty set

$[\mathbf{0} \to A]$
exactly one function:

$f = [$

$\mathbf{1} = \{\bullet\}$
a set with just one element

$[\mathbf{1} \to A]$
exactly $|A|$ functions:

$f_a = [\ \bullet \mapsto a$

From sets to functions

Sets
are about *membership*

- is $x \in A$ or not?

Given A, summarize these answers...

χ_A is the *characteristic function* of A
$\chi_A(x) = 1 \text{ iff } x \in A$

Characteristic examples

χ_A is the *characteristic function* of A
$\chi_A(x) = 1$ iff $x \in A$

Let $E = \{a, b, c, d\}$

$$\chi_\emptyset = \begin{bmatrix} a \mapsto 0 \\ b \mapsto 0 \\ c \mapsto 0 \\ d \mapsto 0 \end{bmatrix} \qquad \chi_{\{a\}} = \begin{bmatrix} a \mapsto 1 \\ b \mapsto 0 \\ c \mapsto 0 \\ d \mapsto 0 \end{bmatrix}$$

$$\chi_{\{b,d\}} = \begin{bmatrix} a \mapsto 0 \\ b \mapsto 1 \\ c \mapsto 0 \\ d \mapsto 1 \end{bmatrix} \qquad \chi_E = \begin{bmatrix} a \mapsto 1 \\ b \mapsto 1 \\ c \mapsto 1 \\ d \mapsto 1 \end{bmatrix}$$

Characteristic functions

χ is a function

- from $\wp E$
- to $[E \to 2]$

it is injective

if $A \neq B$, then $\chi_A \neq \chi_B$

- why?

is it surjective?

- given some $f \in [E \to 2]$,
- is there a set A
  - such that $f = \chi_A$?

From functions to sets

Towards surjectivity

- given some $f \in [E \to 2]$,
- find an A
    - such that $f = \chi_A$

Define X_f

$X_f := \{a : f(a) = 1\}$

Characteristic examples

X_f is the set associated with f

$X_f := \{a : f(a) = 1\}$

Let $E = \{a, b, c, d\}$

$$f = \left[\begin{array}{l} a \mapsto 1 \\ b \mapsto 0 \\ c \mapsto 1 \\ d \mapsto 0 \end{array}\right. \qquad X_f = \{a, c\}$$

$$g = \left[\begin{array}{l} a \mapsto 0 \\ b \mapsto 1 \\ c \mapsto 1 \\ d \mapsto 1 \end{array}\right. \qquad X_g = \{b, c, d\}$$

Back and forth

$X_{\chi_A} = A$

$$\begin{aligned} X_{\chi_A} &= \{a : \chi_A(a) = 1\} \\ &= \{a : a \in A\} \\ &= A \end{aligned}$$

$\chi_{X_f} = f$

$$\begin{aligned} \chi_{X_f}(a) &= 1 \text{ iff } a \in X_f \\ &= 1 \text{ iff } f(a) = 1 \\ &= f(a) \end{aligned}$$

The moral

Sets and Characteristic functions
are two ways of looking at the same thing

$$\wp(E) \cong [E \to 2]$$

Predication

with sets

$$f_S(x, A) = 1 \text{ iff } x \in A$$

with *functions*

$$f_S(x, \chi_A) = \chi_A(x)$$

Multiple predication

with sets

$$f_{VP}(x, R) = \{y : \langle y, x\rangle \in R\}$$

with functions

$$f_{VP}(x, R) = \chi_{\{y:\langle y,x\rangle \in R\}}$$

we would like to turn R into a function...

- that *outputs* another function

Rethinking Relations

Given:

- $E := \{a, b\}$
- $R := \{\langle a, a\rangle, \langle a, b\rangle, \langle b, b\rangle\}$

Question:
Given an object, which subjects go with it?

We write:
$R_y := \{x : \langle x, y\rangle \in R\}$

Rethinking Relations

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Answer:

- $R_a = \{a\}$

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Question:
Given an object, which subjects go with it?

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Answer:

- $R_a = \{a\}$
- $R_b = \{a, b\}$

Relations and functions

A relation R as a function:

$$f_R = \begin{bmatrix} a \mapsto \chi_{R_a} \\ b \mapsto \chi_{R_b} \\ \vdots \end{bmatrix}$$

Constructions unified

subjects

$$f_S(x, g) = g(x)$$

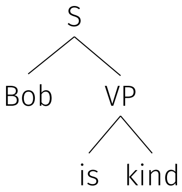
objects

$$f_{VP}(x, g) = g(x)$$

what do they have in common?
they both apply a function to an argument

all constructions are interpreted as function application

Revisiting Predicates



What is the meaning of *is*?

- $[\![kind]\!] \in [E \to 2]$

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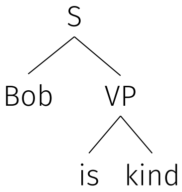

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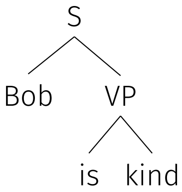

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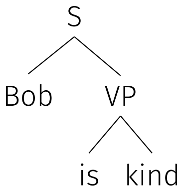

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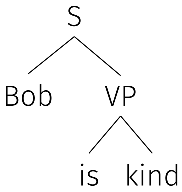

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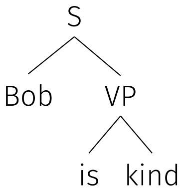

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- we want: $[\![is]\!]\,([\![kind]\!])(b) = [\![kind]\!]\,(b)$
- so... $[\![is]\!]\,(f) = f$

Logical constants

⟦*is*⟧ vs ⟦*kind*⟧

- who is kind depends on the way the world is

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  content words denotations can vary
  function words denotations are fixed

Logical constants

$[\![is]\!]$ vs $[\![kind]\!]$

- who is kind depends on the way the world is
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  content words denotations can vary

  function words denotations are fixed
- $[\![is]\!] = \mathbf{id}$ in *every* model

Interim summary

- the attempt to unify differences
- has not only
  - given us a one-size-fits-all perspective on semantic composition
  - but also a way of investigating meanings of otherwise puzzling words

A slogan

- unification
- via changing perspectives
- can lead to explanation

The meaning of *And*

And, Or, Not

sentences

$$[\![S_1 \text{ and } S_2]\!] = [\![S_1]\!] \;\&\; [\![S_2]\!]$$

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what does and mean?

Unifying sets and truth values

and in truth values $\&$

and in sets $\cap$

Changing notation for unification

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truth values • $x \& y \leq x$

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truth values

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  - if $X \subseteq A$ and $X \subseteq B$

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  - if $X \subseteq A$ and $X \subseteq B$
  - then $X \subseteq A \cap B$

And unified

$[\![\alpha$ *and* $\beta]\!]$
the biggest γ smaller than both α and β

in sets

big and small in terms of *subset*

in truth values

big and small in terms of *implication*

and everything else?

No more sets

Oops
we turned sets into functions

What do big and small mean for functions?

A special case: $[E \to 2]$

- $A \subseteq B$: for every a,
  - if $a \in A$
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    - then $a \in B$
- $\chi_A \leq \chi_B$: for every a
    - if $\chi_A(a) = 1$
    - then $\chi_B(a) = 1$
    - in other words:
      $\chi_A(a) \leq \chi_B(a)$

Generalizing

$$f \leq g \text{ iff for all } x,\ f(x) \leq g(x)$$

- Requires the codomain to be ordered!

Luckily...

- $2 = \{0, 1\}$ is ordered
- therefore
  - $[E \to 2]$ is ordered
  - $[E \to [E \to 2]]$ is ordered
  - $[[E \to 2] \to [E \to 2]]$ is ordered
  - and so on

And and glb

The greatest lower bound of a set A
is the biggest thing smaller than everything else in A

written $\bigwedge A$

or if $A = \{x, y\}$ $x \wedge y$

Claim:
$[\![$*and*$]\!]$ means 'greatest lower bound'

- only works in an ordered domain

What about *Or*

or in truth values $\vee$

or in sets $\cup$

Changing notation for unification

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  - then $A \cap B \subseteq X$

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is the smallest thing bigger than everything else in A

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Claim:
$[\![or]\!]$ means 'least upper bound'

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not in sets $E - \square$

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  - 1 is the biggest number

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- $(E - A) \cup A = E$
- $(E - A) \cap A = \emptyset$

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- $\neg x \wedge x = 0$
- the 'opposite' number
  - 1 is the biggest number
  - 0 is the smallest number

sets
- $(E - A) \cup A = E$
- $(E - A) \cap A = \emptyset$
- the 'opposite' set

What about *Not*

not in truth values $\neg$

not in sets $E - \square$

Changing notation for unification

truth values

- $\neg x \vee x = 1$
- $\neg x \wedge x = 0$
- the 'opposite' number
  - 1 is the biggest number
  - 0 is the smallest number

sets

- $(E - A) \cup A = E$
- $(E - A) \cap A = \emptyset$
- the 'opposite' set
  - E is the biggest set

What about *Not*

not in truth values $\neg$

not in sets $E - \square$

Changing notation for unification

truth values

- $\neg x \vee x = 1$
- $\neg x \wedge x = 0$
- the 'opposite' number
  - 1 is the biggest number
  - 0 is the smallest number

sets

- $(E - A) \cup A = E$
- $(E - A) \cap A = \emptyset$
- the 'opposite' set
  - E is the biggest set
  - $\emptyset$ is the smallest set

No more sets

Oops
we turned sets into functions

What does opposite mean for functions

A special case: $[E \to 2]$

- the biggest set is E
  - so the biggest function is χ_E
- the smallest set is $\emptyset$
  - so the smallest function is $\chi_\emptyset$
- the opposite set of A is $E - A$
  - the opposite function should 'flip' all 0s and 1s

Generalizing

Biggest

$$1_{A \to B}(a) = 1_B$$

Smallest

$$0_{A \to B}(a) = 0_B$$

Opposite

$$(\neg f)(a) = \neg(f(a))$$

Not and complement

The complement of something
is its opposite

- the glb of something and its opposite is the smallest thing
- the lub of something and its opposite is the biggest thing

Claim:
⟦*not*⟧ means 'complement'

- only works in an ordered domain

Interim summary

Notational history

- I wrote 1 for *true*
- and 0 for *false*
- because *true* is the biggest truth value
- and *false* the smallest
- with respect to negation:
  - $\neg b \vee b = \mathit{true}$
  - $\neg b \wedge b = \mathit{false}$

Any ordered domain

can be operated on booleanly:

- laugh and praise Mary
- praise or criticize

Boolean Lattices

Boolean lattice

- a partially ordered set
- with meets
- with joins
- which is bounded
- which is distributive
- which is complemented

Partial orders

A set A is partially ordered if

- there is a binary relation (written $\leq$) over A
- which is *reflexive* : for all x,
  - $x \leq x$
- which is *asymmetric* : for all $x \neq y$,
  - not both $x \leq y$ and $y \leq x$
- which is *transitive* : forall x, y, z,
  - if $x \leq y$
  - and $y \leq z$
  - then $x \leq z$

Meets and Joins

A partial order has *meets* and *joins* if

- for any elements $x, y \in A$
  - $x \vee y$ is defined (the smallest thing bigger than x and y)
  - $x \wedge y$ is defined (the biggest thing smaller than x and y)

A lattice
is a partial order with meets and joins

Boundedness

A lattice is *bounded* if

- it has a greatest element 1
- and a smallest element 0

Distributivity

A lattice is *distributive* if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Complementation

A bounded lattice has complements if

- for every element $a \in A$
  - its opposite element exists

Building Boolean lattices

if B is a boolean lattice, then for any set A

$[A \to B]$ is a boolean lattice

- $f \leq g$ iff for every a, $f(a) \leq g(a)$
- $(f \wedge g)(a) := f(a) \wedge g(a)$
- $1_{A \to B}(a) := 1_B$
- $(\neg f)(a) := \neg(f(a))$

Summary

The denotation domains of natural language expressions

- are *functions*
- are *boolean lattices*

except for *E*...

more on this next time!