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Topological Phases and Anyons — Problem Set 3

Fall 2016/2017

Due: This problem set is due on **Friday, 27. Jan 2017**.

Remark: Let A, B be two operators. Their *commutator* is defined as $[A, B] \equiv AB - BA$, while their *anticommutator* is defined as $\{A, B\} \equiv AB + BA$.

6. Fermionic creation and annihilation operators *1+2 points*

In previous problems, we have looked at the wave function $\psi(x)$ of a single particle, where the quantity $|\psi(x)|^2$ corresponds to the classical probability of finding the particle at position x.

We now take a look at multiple particles, which are described by a combined many-particle wave function $\psi(x_1, x_2, \dots)$, with the interpration that $|\psi(x_1, x_2, \dots)|^2$ is the probability of finding the first particle at position x_1, the second at x_2, and so on. If the particles are *indistinguishable*, then the classical probability will not change if we interchange, say, the first and second particle $|\psi(x_1, x_2, \dots)|^2 = |\psi(x_2, x_1, \dots)|^2$. However, that still leaves a choice of complex phase. *Fermionic particles* are defined by the relation $\psi(x_1, x_2, \dots) = -\psi(x_2, x_1, \dots)$, i.e. their combined wave function changes sign upon interchange two particles. (Other possibilities are *bosonic* and, in two space dimensions, *anyonic* particles.)

In this problem, we look at the *fermionic creation* and *annihilation operators*, which are a convenient way of organizing a Hilbert space of fermionic particles. It turns out that allowing a variable number of particles is particularly convenient.

Let H be a complex Hilbert space and $(c_i)_{i \in I}$ a family of (bounded) operators acting on this Hilbert space. We denote their hermitian conjugates by $c_i^\dagger$, so that

$$\langle c_i(v), w \rangle = \langle v, c_i^\dagger(w) \rangle, \quad \text{for all } v, w \in H, i \in I. \tag{1}$$

The c_i are called *annihilation operators* and the $c_i^\dagger$ are called *creation operators* if they satisfy the following relations, called *canonical anticommutation relations*:

$$\{c_i, c_j\} = 0, \quad \{c_i^\dagger, c_j^\dagger\} = 0, \quad \{c_i, c_j^\dagger\} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for all } i, j \in I. \tag{2}$$

The use of anticommutation relations instead of commutation relations encodes the (-1) sign that occurs when interchanging two fermionic particles.

Furthermore, the product

$$n_i := c_i^\dagger c_i \tag{3}$$

is called the *particle number operator*.

(a) Show that the particle number operators satisfy $n_i^2 = n_i$ for all $i \in I$. This implies that its eigenvalue can only be 1 or 0. It counts whether a fermion is present or absent at position j.

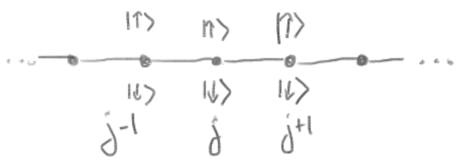
(b) Show that the particle number operators commute, i.e. $[n_i, n_j] = 0$ for all $i, j \in I$. This implies that they can be diagonalized simultaneously.

A Hilbert space of minimal size that admits a representation of a family of annihilation and creation operators is called the *Fock space* for this family. In the next problem, we will look at a particular representation in terms of a spin chain.

7. Jordan-Wigner transformation *1+4+3 points*

In this problem, we will look at a lattice model of spins and find that we can understand it in terms of quasiparticle excitations that correspond to fermionic particles. These particles are created by string operators.

Consider the infinite one-dimensional lattice, $\mathbb{Z}$. To each lattice site, we associate a two-dimensional vectors space spanned by a spin-up $|\uparrow\rangle$ and a spin-down vector $|\downarrow\rangle$. The Hilbert space of physical states is essentially given by the tensor product of all these vector spaces. This space is spanned by vectors of the form $\ldots|\uparrow\rangle\otimes|\uparrow\rangle\otimes|\downarrow\rangle\ldots$, which we abbreviate $|\ldots\uparrow\uparrow\downarrow\ldots\rangle$. (For simplicity, we only consider the separable Hilbert space spanned by those states where all but finitely many lattice sites are in the spin-down state.) Here is an illustration of the system:



At each lattice site j, we have Pauli matrix operators $\sigma_j^x, \sigma_j^y, \sigma_j^z$ which act on the spin state on this site. For example, σ_j^x corresponds to a spin flip while σ_j^z will multiply the state with (-1) if the spin-state is down, e.g.

$$\sigma_j^x|\ldots\downarrow\downarrow\underbrace{\downarrow}_{j}\downarrow\downarrow\ldots\rangle = |\ldots\downarrow\downarrow\underbrace{\uparrow}_{j}\downarrow\downarrow\ldots\rangle \tag{4}$$

$$\sigma_j^z|\ldots\downarrow\downarrow\underbrace{\downarrow}_{j}\downarrow\downarrow\ldots\rangle = -|\ldots\downarrow\downarrow\underbrace{\downarrow}_{j}\downarrow\downarrow\ldots\rangle. \tag{5}$$

(These operations are extended linearly to the whole Hilbert space.)

Note that spin operators on different lattice sites always commute, $[\sigma_j^{x,y,z}, \sigma_k^{x,y,z}] = 0$ if $j \neq k$, while spin operators on the same site satisfy the anticommutation relations of the Pauli matrices, for example $\{\sigma_j^x, \sigma_j^y\} = 0$.

Now, for each lattice site j, we can define an operator

$$c_j := \left(\prod_{l<j, l\in\mathbb{Z}} (-\sigma_l^z) \right) \cdot \frac{1}{2}(\sigma_j^x - i\sigma_j^y). \tag{6}$$

The infinite product is taken over a "string" of lattice sites extending from left infinity to (but not including) the lattice site j. (Since we restrict our attention to the Hilbert space where all but finitely many lattice sites are in the spin-down state, this product is actually finite on every state.)

(a) From the formula for the operator c_j, determine a similar formula for the hermitian conjugate of the operator, $c_j^\dagger$. (Hint: The Pauli matrices are hermitian, $(\sigma^{x,y,z})^\dagger = \sigma^{x,y,z}$.)

(b) Show that the operators c_j represent fermions, i.e. that they satisfy the canonical anticommutation relations as defined in the previous problem. In particular, show that they *anticommute* for different lattice sites, e.g. $\{c_j, c_k^\dagger\} = \delta_{jk}$.

(c) Consider the Hamiltonian

$$H = -\frac{J_{xy}}{4} \sum_{j\in\mathbb{Z}} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y). \tag{7}$$

Show that in terms of fermionic operators, it can be expressed as

$$H = -\frac{J_{xy}}{2} \sum_{j\in\mathbb{Z}} (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j). \tag{8}$$

(Hint: The identities $\sigma_j^z c_j = c_j$ and $\sigma_j^z c_j^\dagger = -c_j^\dagger$ may prove useful.)

In other words, it is possible to map a spin system in one dimension to a system of fermions. This procedure also known as the *Jordan-Wigner transformation.* The new Hamiltonian (8) has the virtue of being quadratic in the fermion operators, which means that it can be solved exactly, because Hamiltonians of this type correspond to non-interacting fermionic particles.