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Topological Phases and Anyons — Problem Set 2

Fall 2016/2017

Due: This problem set is due on **Friday, 20. Jan 2017**.

Remark: Remember that $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ denotes the vector whose components are Pauli matrices. If $\vec{a} \in \mathbb{R}^3$ is a vector with real components, then the "scalar product" $\vec{a}\vec{\sigma} = \vec{a} \cdot \vec{\sigma} = a_x\sigma^x + a_y\sigma^y + a_z\sigma^z \in \mathbb{C}^{2\times 2}$ is a 2×2 matrix with complex entries.

Remark: We denote the 2×2 identity matrix by $\mathbf{1}$.

4. Pauli matrices and spatial rotations *2+2+3 points*

In this problem, we take a closer look at the connection between the Pauli matrices and spatial rotations.

(a) Let $\vec{a}, \vec{b} \in \mathbb{R}^3$ be two three-dimensional vectors. Prove the matrix identity

$$(\vec{a}\vec{\sigma})(\vec{b}\vec{\sigma}) = (\vec{a} \cdot \vec{b})\mathbf{1} + i(\vec{a} \times \vec{b})\vec{\sigma} \tag{1}$$

where $\vec{a} \cdot \vec{b}$ denotes the scalar product and $\vec{a} \times \vec{b}$ the vector product of the two vectors.

(b) Let $A \in \mathbb{C}^{2\times 2}$ be a hermitian matrix ($\overline{A} = A^T$) which is traceles ($\operatorname{tr} A = 0$). Show that there exists a unique vector $\vec{a} \in \mathbb{R}^3$ such that $A = \vec{a}\vec{\sigma}$.

(c) Show that for any unitary matrix $U \in \mathbb{C}^{2\times 2}$, there exists a unique rotation matrix $R \in SO(3)$ such that

$$U(\vec{a}\vec{\sigma})U^\dagger = (R\vec{a})\vec{\sigma}, \quad \text{for all } \vec{a} \in \mathbb{R}^3, \tag{2}$$

where $R\vec{a}$ denotes the vector obtained by applying the rotation R to the vector $\vec{a}$. Do not forget to show that the rotation matrix preserves orientation.

(Hint: Use (a) and (b) to construct the matrix R and prove that it preserves scalar products and vector products.)

The last construction is the famous double cover $SU(2) \to SO(3)$, also known as the representation of spatial rotations by quaternions. Essentially, the matrices $\mathbf{1}, i\sigma^x, i\sigma^y, i\sigma^z$ are a representation of Hamilton's *quaternions* $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$. However, in quantum mechanics, the Pauli matrices are more commonly used than the quaternions, probably because the former are hermitian, while the latter must be antihermitian.

5. Chern numbers and two-dimensional winding numbers *4+3+2 points*

In the lecture, we have looked at Bloch Hamiltonians of the form $H(k) = \mathbf{d}(k)\vec{\sigma}$ where $\mathbf{d} : T \to \mathbb{R}^3$ is a smooth map from the Brillouin zone T to the space of three-dimensional vectors $\mathbb{R}^3$. If the underlying lattice model is two-dimensional, then the Brillouin zone is the two-dimensional torus, $T = S^1 \times S^1$.

Let us restrict our attention to *gapped* Bloch Hamiltonians, i.e. there is a gap size $\epsilon > 0$ such that $\|\mathbf{d}(k)\| \geq \epsilon$. Then, the unit vector $\mathbf{n}(k) = \mathbf{d}(k)/\|\mathbf{d}(k)\|$ defines a smooth map $\mathbf{n} : T \to S^2$ from the torus to the sphere. The following integral measures how often this unit vector "wraps" around the sphere:

$$W = \frac{1}{4\pi} \int_T dk_1 dk_2 \left(\frac{\partial \mathbf{n}}{\partial k_1} \times \frac{\partial \mathbf{n}}{\partial k_2} \right) \cdot \mathbf{n}. \tag{3}$$

This is the higher-dimensional analogue of the *winding number* of a path in the plane around the origin.

In the lecture, we have also considered the eigenvalues and eigenvectors of such Bloch Hamiltonians. The eigenvalues are readily shown to be $E(k) = \pm\|\mathbf{d}(k)\|$, because the square $H(k)^2 = \|\mathbf{d}(k)\|^2 \mathbf{1}$ is a diagonal matrix. For each momentum k_0, let $\psi(k_0) = (u, v)^T \in \mathbb{C}^2$ be a normalized eigenvector for the positive eigenvalue $+\|d(k_0)\|$. It also satisfies $\mathbf{n}(k_0)\psi(k_0) = \psi(k_0)$. Locally, i.e. in a small neighborhood of the point k_0, we can extend this eigenvector to a smooth family $\psi(k)$ of normalized eigenvectors of the operators $H(k)$ (though this may not be possible globally). In the lecture, this has allowed us to define the *Berry curvature*

$$\mathcal{F} = i[\langle \partial_{k_1}\psi | \partial_{k_2}\psi \rangle - \langle \partial_{k_2}\psi | \partial_{k_1}\psi \rangle] \tag{4}$$

at every point. We have shown that it is independent of the choice of extension, hence, we were also able to define the *Chern number*

$$C = -\frac{1}{2\pi} \int_T dk_1 dk_2 \, \mathcal{F}. \tag{5}$$

In this problem, we want to show that the two integrals are the same, $W = C$. To that end, let $\mathcal{S} = (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}) \cdot \mathbf{n}$ denote the integrand for the winding number.

(a) Consider the special case where there exists a point k_0 such that $\mathbf{n}(k_0) = (0, 0, 1)^T$. Show that a corresponding normalized eigenvector is given by $\psi(k_0) = (u(k_0), v(k_0)) = (1, 0)$. Use the eigenvalue equation, $(\mathbf{n}\vec{\sigma})\psi = \psi$, and normalization, $\langle \psi | \psi \rangle = 1$, to prove that $\mathcal{F} = -2\mathcal{S}$ at such a point k_0.

(b) Let U be any unitary matrix. Use the results from previous exercise to show that the family of Hamiltonians defined by $\tilde{H}(k) = UH(k)U^\dagger$ can be written in the form $\tilde{H}(k) = \|\mathbf{d}(k)\|\tilde{\mathbf{n}}(k)\vec{\sigma}$ with a unit vector $\tilde{\mathbf{n}}(k)$. Also show that the associated integrands are equal everywhere, that is $\tilde{\mathcal{F}} = \mathcal{F}$ and $\tilde{\mathcal{S}} = \mathcal{S}$.

(c) For each point k_0, use the eigenvector $\psi(k_0)$ of $H(k_0)$ to construct a unitary matrix U with the property that $\tilde{\mathbf{n}}(k_0) = (0, 0, 1)^T$. Comining (a) and (b), it now follows that $\mathcal{F} = -2\mathcal{S}$ everywhere, which concludes the proof.