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Topological Phases and Anyons — Problem Set 1

Fall 2016/2017

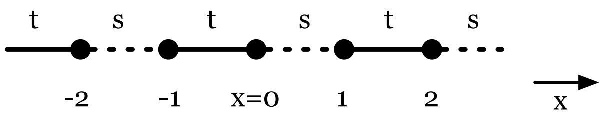
Due: This problem set is due on **Friday, 13. Jan 2017**.

Remark: A function $\psi : \mathbb{Z} \to \mathbb{C}$ is called *square integrable* iff $\sum_{x\in\mathbb{Z}} |\psi(x)|^2 < \infty$. The square integrable functions form a Hilbert space with scalar product $\langle\phi|\psi\rangle = \sum_{x\in\mathbb{Z}} \overline{\phi(x)}\psi(x)$ and norm $\|\psi\| = \sqrt{\langle\psi|\psi\rangle}$.

Remark: Similarly, a measurable function $\psi : \mathbb{R}^d \to \mathbb{C}^2$ is called *square integrable* iff $\int_{\mathbb{R}^d} dx \sum_{\alpha=1,2} |\psi_\alpha(x)|^2 < \infty$. Again, these functions form a Hilbert space with scalar product $\langle\phi|\psi\rangle = \int_{\mathbb{R}^d} dx \sum_{\alpha=1,2} \overline{\phi_\alpha(x)}\psi_\alpha(x)$.

1. Su-Schrieffer-Heeger (SSH) model on a lattice *2+2+1+1 Points*

The Su-Schrieffer-Heeger (SSH) model is a tight-binding model for electrons on a one-dimensional lattice. Electrons can hop from each lattice site to its nearest neighbors, but the hopping amplitudes are alternating. Here is a picture:



This model can be formalized by the following Hamiltonian:

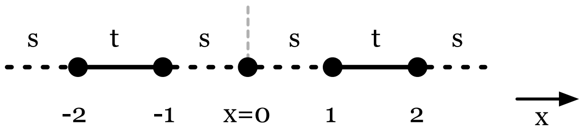
$$(H^{SSH}_{s,t}\psi)(x) = \begin{cases} t\psi(x-1) + s\psi(x+1), & \text{if } x \text{ even} \\ s\psi(x-1) + t\psi(x+1), & \text{if } x \text{ odd.} \end{cases} \tag{1}$$

Here $s, t > 0$ are the alternating hopping amplitudes, x denotes a lattice site and $\psi(x) \in \mathbb{C}$ denotes the probability amplitude of the electron at that lattice site. We consider an infinite lattice, that is $x \in \mathbb{Z}$.

(a) Let ψ be any square integrable wave function. Prove that $\|H^{SSH}_{s,t}\psi\| \geq |s-t|\|\psi\|$.

(This means that the spectrum of the operator $H^{SSH}_{s,t}$ has a gap of diameter at least $2|s-t|$ around the origin.)

Now, consider the situation where two SSH models are "joined" together at the lattice site $x = 0$, but such that the alternating pattern is flipped. Picture:



In other words, consider the following Hamiltonian:

$$(H\psi)(x) = \begin{cases} s\psi(-1) + s\psi(1), & \text{if } x = 0 \\ (H^{SSH}_{s,t}\psi)(x), & \text{if } x > 0 \\ (H^{SSH}_{t,s}\psi)(x), & \text{if } x < 0. \end{cases} \tag{2}$$

(b) If $s \neq t$, then, up to a constant prefactor, there is exactly one bounded function $\psi_0 : \mathbb{Z} \to \mathbb{C}$ that satisfies $H\psi_0 = 0$. Give an explicit formula for the amplitude $\psi_0(x)$ and note that it is square integrable.

(c) Apparently, joining two gapped models has suddenly produced a model with a state inside the gap. Explain this phenomenon heuristically in terms of topological invariants of the SSH Hamiltonian.

(d) Consider the special case $s = 0$. Draw a picture of the hopping amplitudes and use it to determine (sketch) the exact spectrum of the (bounded, self-adjoint) operators $H^{SSH}_{s=0,t}$ and H for this case.

2. (Optional) Pauli matrices *3+1+1+1+1 Points*

(This problem is optional. If you are already familiar with the Pauli matrices, you can skip this problem.)

Consider the complex vector space $\mathbb{C}^2$. The matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

are called the *Pauli matrices.* The following exercises explore their properties.

(a) Consider the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{4}$$

Prove the following identities by calculation:

$$U^{-1}\sigma^y U = \sigma^z, \quad U^{-1}\sigma^z U = \sigma^x, \quad U^{-1}\sigma^x U = \sigma^y. \tag{5}$$

(b) Prove that $\sigma^x\sigma^y = i\sigma^z$ and $\sigma^y\sigma^x = -i\sigma^z$ by calculation.

(In particular, the Pauli matrices *anticommute*, $\sigma^x\sigma^y = -\sigma^x\sigma^y$.)

Use (a) to conclude that if you have an algebraic identity of Pauli matrices, then applying a cyclic permutation of the letters, e.g. $(x, y, z) \to (y, z, x)$ will give another identity. For instance, we can conclude that $\sigma^y \sigma^z = i\sigma^x$ also holds.

(c) Prove that $\mathbf{1}_{2\times 2} = (\sigma^z)^2 = (\sigma^x)^2 = (\sigma^y)^2$.

(d) Let $\vec{a} = (a_1, a_2, a_3)^T \in \mathbb{R}^3$ denote a column vector and let $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ denote a vector whose entries are the Pauli matrices. Then, the scalar product $\vec{a}\cdot\vec{\sigma} = a_1\sigma^x + a_2\sigma^y + a_3\sigma^z$ is a matrix. Show that this matrix satisfies $(\vec{a}\cdot\vec{\sigma})^2 = |\vec{a}|^2 \mathbf{1}_{2\times 2}$.

(e) Prove that the Pauli matrices are hermitian matrices, i.e. $\langle \phi | \sigma^y \psi \rangle = \langle \sigma^y \phi | \psi \rangle$ where $\langle \phi | \psi \rangle = \sum_{\alpha=\uparrow,\downarrow} \overline{\phi_\alpha} \psi_\alpha$ is the scalar product on the complex vector space $\mathbb{C}^2$.

3. Massive Dirac Hamiltonian and domain walls in 2D *4 Points*

(Familiarity with the Pauli matrices is very helpful for solving this problem.)

Lattice models can be difficult to solve. It is often convenient to make a *continuum approximation*, where we consider only wave functions that vary very slowly on length scales comparable to the distance between lattice sites. This allows us to replace the discrete difference by a derivative.

The simplest two-dimensional lattice model of a topological insulator is probably the Haldane model. In the continuum approximation, it can be described by the massive Dirac Hamiltonian. In this case, the slowly varying wave functions are modeled by a pair of continuously differentiable functions $\psi_\uparrow, \psi_\downarrow : \mathbb{R}^2 \to \mathbb{C}$. It is convenient to collect them in a column vector $\psi = (\psi_\uparrow, \psi_\downarrow)^T$. Then, the continuum Hamiltonian reads

$$(H\psi)(x, y) = \left[-i \left(\sigma^x \frac{\partial}{\partial x} + \sigma^y \frac{\partial}{\partial y} \right) + m(x, y)\sigma^z \right] \psi(x, y), \tag{6}$$

where $\sigma^x, \sigma^y, \sigma^z$ are the Pauli matrices acting on the column vector, and $m(x, y) \in \mathbb{R}$ is a parameter profile commonly interpreted as a "mass".

When the mass profile is constant, $m(x, y) \equiv \pm m_0 \neq 0$, it is not difficult to show that this Hamiltonian has a spectral gap, $\langle H\psi | H\psi \rangle \geq |m_0| \langle \psi | \psi \rangle$.

(a) Let $m_0 > 0$ be a fixed constant. Consider now a smooth mass profile $m(x, y) \equiv m(x)$ that is constant in y-direction, but interpolates between the two values $m(x \to -\infty) = -m_0$ and $m(x \to +\infty) = +m_0$ in x-direction. This is also called a *domain wall.*

We want to find solutions to the eigenvalue equation $H\psi = E\psi$. The Hamiltonian is invariant under translations in y-direction, hence we can make the ansatz $\psi(x, y) = e^{ik_y y}\tilde{\psi}(x)$ where $k_y \in \mathbb{R}$ denotes momentum. For each momentum, find a square integrable function $\tilde{\psi}(x)$ such that this ansatz solves the eigenvalue equation with eigenvalue $E = k_y$.

(In particular, these energies lie inside the spectral gap of a constant mass profile.)

(Hint: Use the unitary matrix U from the previous problem (a) and consider the Hamiltonian $\tilde{H} = U^{-1}HU$ instead. This step will essentially permute the Pauli matrices, $(\sigma^x, \sigma^y, \sigma^z) \to (\sigma^y, \sigma^z, \sigma^x)$. After you have done that, you should be able to find an eigenfunction for the Hamiltonian $\tilde{H}$ in the form $\tilde{\psi}(x) = (f(x), 0)^T$ by proceeding in a way similar to the 1D domain wall discussed in the lecture. Use the matrix U to find an eigenfunction for the original Hamiltonian.)