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Advanced Statistical Physics - Problem Set 9

Summer Term 2025

Due Date: Thursday, June 05, 17:00. Hand in tasks marked with * via Moodle

The first problem explores the role of domain wall formation and finite-size effects for the Ising phase transition. In the second problem you compute the size of a Cooper pair, which will be used later on to understand why fluctuation effects are not important for the normal metal to superconductor phase transition.

1. Domain wall picture of the Ising phase transition

2 + 2 + 2 + 1 + 2 Points

The thermodynamic limit is a mathematical necessity to observe the singular behavior of susceptibilities or correlations lengths. For systems of finite size, the free energy can be expressed as a finite order polynomial of the order parameter and its derivatives. Thus, no phase transition in the mathematical sense can be observed. The aim of this exercise is to show that if one includes dynamics, the thermodynamic limit is not necessary to observe phase transitions.
Consider a two-dimensional lattice with periodic boundary conditions of N spins with interaction constant J which is in the ordered phase for $T < T_c$. Here, either the state with magnetization $m = -1$ or $m = +1$ is spontaneously realized. To switch between them, the formation of a domain wall is needed as shown in the figure on the next page.
The aim of the task is to estimate the critical temperature T_c and the number of spins N that is actually needed in order to get a physically stable thermodynamic phase.

a) Verify that the change in free energy ΔF_{wall} of the domain wall of length L can be estimated as

$$\Delta F_{\text{wall}} = L\left(2J - k_B T \ln 3\right) \ .$$

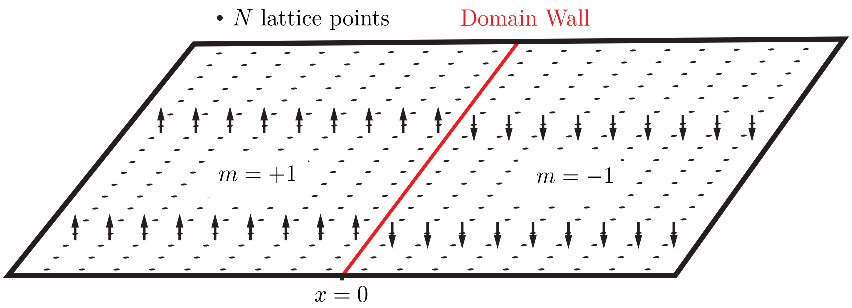
Hint: To calculate the internal energy, estimate the energy that is needed to flip one spin. The entropy is given by $S = k_B \ln W$, with W being the number of possible realizations of domain walls with length L. At each lattice site, there are three possible directions for the domain wall.

b) Argue that a phase transition occurs at $\Delta F_{\text{wall}} = 0$. Determine the critical temperature and compare to the Onsager solution for the 2D Ising model

$$k_B T_c^{\text{Onsager}} = \frac{2J}{\ln\left(1+\sqrt{2}\right)} \ .$$

Comment on the validity of your estimation.

c) Now argue that for the formation of a domain wall separating the system into two distinct regions we must have $L \geq \sqrt{N}$, and determine ΔF_{wall} for $T = T_c/2$ and $L = \sqrt{N}$.



Domain wall separating two regions of different magnetization in the ordered phase.

d) Finally, assume that the dynamics follows the Arrhenius law

$$\tau = \tau_0 \cdot e^{\beta \Delta F_{\text{wall}}} ,$$

with $\tau_0 = 10^{-12}\,\text{s}$ being the characteristic microscopic spin-flip time. Again working with $T = T_c/2$ estimate the number of spins, which are necessary to spontaneously flip the magnetization within the lifetime of the universe $\tau = 4,3 \cdot 10^{17}\,\text{s}$.

*2. Cooper pair size

2+4+1 Points

a) The Cooper pair wave function can be expanded in a plane-wave basis as

$$\psi(\boldsymbol{r}) = \sum_{\boldsymbol{k}} g(\boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{r}} ,$$

with $g(\boldsymbol{k})$ being the amplitude of finding an electron in state with momentum $\boldsymbol{k}$ and another one in a state with momentum $-\boldsymbol{k}$, and $\boldsymbol{r}$ being the relative coordinate (distance between the electrons) of the Cooper pair. Note that $g(\boldsymbol{k}) = 0$ for $k < k_F$. Start with the definition of the mean square radius

$$R^2 = \frac{\int d\boldsymbol{r}\, r^2 |\psi(\boldsymbol{r})|^2}{\int d\boldsymbol{r}\, |\psi(\boldsymbol{r})|^2} ,$$

and show that it can be reexpressed as

$$R^2 = \frac{\sum_{\boldsymbol{k}} |\nabla_{\boldsymbol{k}}\, g(\boldsymbol{k})|^2}{\sum_{\boldsymbol{k}} |g(\boldsymbol{k})|^2} .$$

b) By turning the sums into integrals and by using the definition of the density of state, show that the expression for the mean square radius becomes

$$R^2 \simeq \frac{\left(\frac{\partial \xi}{\partial k}\right)^2_{\xi=0} \int\limits_0^\infty d\xi \left(\frac{\partial g(\xi)}{\partial \xi}\right)^2}{\int\limits_0^\infty d\xi\, g(\xi)^2} .$$

Further, use that $g(\xi) \propto 1/(\Delta + 2\xi)$ to show that the mean square radius of a Cooper pair is given by

$$R = \frac{2}{\sqrt{3}} \frac{\hbar v_F}{\Delta} ,$$

with v_F being the Fermi velocity, and Δ being the binding energy of the Cooper pair relative to the Fermi surface.

c) Insert realistic values for v_F and Δ and estimate the size of the Cooper pair.