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Advanced Statistical Physics - Problem Set 7

Summer Term 2025

Due Date: Thursday, May 22, 17:00., Hand in tasks marked with * via Moodle.

The first problem is a mathematical problem, which will help you to get familiar with Fourier transformations. The second problem illustrates the use of the mean field approximation in the description of phase transitions.

1. Fourier transform *2 + 3 Points*

a) The Fourier transform of a real field $\psi(x)$ is defined as

$$\psi(x) = \frac{1}{\sqrt{L^d}} \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \psi_{\boldsymbol{k}}.$$

Show that the inverse transformation is given by

$$\psi_{\boldsymbol{k}} = \frac{1}{\sqrt{L^d}} \int d^d x \; e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \psi(x) = \alpha_{\boldsymbol{k}} + i\beta_{\boldsymbol{k}},$$

and further deduce that $\alpha_{\boldsymbol{k}} = \alpha_{-\boldsymbol{k}}$ and $\beta_{\boldsymbol{k}} = -\beta_{-\boldsymbol{k}}$.

b) Explicitly derive the Fourier transform of the Landau-Ginzburg Hamiltonian

$$\mathcal{H} = \int d^d x \left[\frac{a\tau}{2}\psi^2(x) + \frac{u}{4}\psi^4(x) + \frac{c}{2}(\nabla\psi)^2 - h(x)\psi(x) \right].$$

*2. Mean field approximation *2 + 2 + 3 + 2 Points*

A crystalline film (simple cubic) is obtained by depositing a finite number of layers n. Each atom has a three component (Heisenberg) spin, and they interact through the Hamiltonian

$$\mathcal{H} = -\sum_{\alpha=1}^{n} \sum_{\langle i,j \rangle} J_H \boldsymbol{s}_i^\alpha \cdot \boldsymbol{s}_j^\alpha - \sum_{\alpha=1}^{n-1} \sum_i J_V \boldsymbol{s}_i^\alpha \cdot \boldsymbol{s}_i^{\alpha+1} \;,$$

where the unit vector $\boldsymbol{s}_i^\alpha$ indicates the spin at site i in the α-th layer, J_H and J_V are the interaction strength, and the subscript $\langle i,j \rangle$ indicates that the spin at i interacts with its four nearest neighbors, indexed by j on the square lattice on the layer.
A mean field Hamiltonian is obtained by expressing the spins at $\boldsymbol{s}_i^\alpha = \langle \boldsymbol{s}_i^\alpha \rangle + (\boldsymbol{s}_i^\alpha - \langle \boldsymbol{s}_{\boldsymbol{i}}^{\boldsymbol{\alpha}} \rangle)$ and assuming that the fluctuations around the expectation values are small. The mean field Hamiltonian can be written in the form

$$\mathcal{H}_0 = -\sum_{\alpha=1}^{n} \sum_i \boldsymbol{h}^\alpha \cdot \boldsymbol{s}_i^\alpha.$$

(a) Show that the partition function Z_0 for the mean field Hamiltonian $\mathcal{H}_0$ can be written as

$$Z_0 = \prod_\alpha \left(\frac{4\pi \sinh \beta h_\alpha}{\beta h_\alpha} \right) ,$$

where $h_\alpha = |\boldsymbol{h}^\alpha|$ and N is the number of sites in each layer.

Hint: Express the $\boldsymbol{h}^\alpha \cdot \boldsymbol{s}_i^\alpha$ in spherical coordinates and integrate over the angles.

(b) Argue that the magnetization for the mean field Hamiltonian $\mathcal{H}_0$ is given by

$$\boldsymbol{m}^\alpha = \hat{\boldsymbol{h}}^\alpha \left[\coth \beta h_\alpha - \frac{1}{\beta h_\alpha} \right] \equiv \hat{h}^\alpha \mathcal{L}(\beta h_\alpha),$$

where $\hat{\boldsymbol{h}}^\alpha$ is the unit vector in the direction of $\boldsymbol{h}^\alpha$.

(c) By comparing $\mathcal{H}$ and $\mathcal{H}_0$, argue that the self-consistency condition for the mean field ansatz is

$$\boldsymbol{h}^\alpha = 4J_H \boldsymbol{m}^\alpha + J_V \left(\boldsymbol{m}^{\alpha+1} + \boldsymbol{m}^{\alpha-1} \right).$$

Assume that the mean fields $\boldsymbol{h}^\alpha$ in all layers are parallel to each other.

Use the result from (b) and the self-consistency condition to find the critical temperature and the behavior of the magnetization in the limit $n \to \infty$.

Hint: In this limit all layers are equal. You may use the expansion $\mathcal{L}^{-1}(m) = 3m + 9m^3/5 + \mathcal{O}(m^5)$.

(d) Show that for finite number of layers, the self-consistency condition results in linearized recursion relations

$$3m_\alpha = 4\beta J_H m_\alpha + \beta J_V \left(m_{\alpha+1} + m_{\alpha-1} \right)$$

with boundary conditions $m_0 = m_{n+1} = 0$.

Solve these recursion relations to show that for $n \gg 1$ the critical temperature depends on the number of layers n, as

$$k_B T_c(n) \approx k_B T_c(\infty) - J_V \pi^2 / (3n^2).$$