
Advanced Statistical Physics - Problem Set 5

Summer Term 2025

Due Date: Thursday, May 8, 17:00. Hand in tasks marked with * via Moodle.

1. BCS as a mean-field theory

4+4+4+4 Points

In this problem we will consider electrons interacting through the BCS interaction, which is non-zero in a shell around the Fermi surface. We consider the following Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \sum_{\mathbf{k}\mathbf{k}'} V(\mathbf{k}, \mathbf{k}') c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} ,$$

where

$$V(\mathbf{k}, \mathbf{k}') = \begin{cases} -V/\Omega, & |\xi_{\mathbf{k}}|, |\xi_{\mathbf{k}'}| \leq \hbar\omega_D, \\ 0, & \text{otherwise} \end{cases} .$$

Here, $\xi_{\mathbf{k}} = \hbar^2 \mathbf{k}^2 / 2m - E_F$, and ω_D denotes the Debye frequency. Motivated by the condensation into momentum space pairs $(\mathbf{k}\uparrow, \mathbf{k}\downarrow)$, we introduce a mean field expectation value $\langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle$, which is called condensation amplitude when computed with respect to the BCS ground state $|\phi\rangle$.

- a) * Proceeding analogously to Hartree-Fock mean-field theory as shown in the lectures, convince yourself that the mean-field decoupling of the Hamiltonian yields

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \Delta_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger - \sum_{\mathbf{k}} \Delta_{\mathbf{k}}^* c_{\mathbf{k}\downarrow} c_{-\mathbf{k}\uparrow} - \sum_{\mathbf{k}\mathbf{k}'} V(\mathbf{k}, \mathbf{k}') \langle c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger \rangle \langle c_{\mathbf{k}'\downarrow} c_{-\mathbf{k}'\uparrow} \rangle ,$$

with the self-consistency condition given by

$$\Delta_{\mathbf{k}} = - \sum_{\mathbf{k}'} V(\mathbf{k}, \mathbf{k}') \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} \rangle .$$

- b) * Show that the Hamiltonian can be diagonalized by the following unitary transformation

$$\begin{pmatrix} \gamma_{\mathbf{k}\uparrow} \\ \gamma_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix} = \begin{pmatrix} \cos \theta_{\mathbf{k}} & \sin \theta_{\mathbf{k}} \\ \sin \theta_{\mathbf{k}} & -\cos \theta_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow}^\dagger \end{pmatrix} ,$$

such that

$$H = \sum_{\mathbf{k}\sigma} E(\mathbf{k}) \gamma_{\mathbf{k}\sigma}^\dagger \gamma_{\mathbf{k}\sigma} + \text{constant}.$$

Derive an expression for $E(\mathbf{k})$.

- c) Show that at zero temperature the self-consistency condition is equivalent to the BCS gap equation. Next, use the self-consistency condition to find the transition temperature T_c for the case $T > 0$.

Hint: Use the unitary transformation obtained in part (b) to rewrite the product $c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}$ in terms of $\gamma_{\mathbf{k}\sigma}^\dagger$ and $\gamma_{\mathbf{k}\sigma}$. Next, use that $\langle \gamma_{\mathbf{k}\sigma}^\dagger \gamma_{\mathbf{k}'\sigma'} \rangle = f_{\mathbf{k}} \cdot \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'}$, with $f_{\mathbf{k}}$ being the Fermi function as defined on problem set 4. In the case $T > 0$, you will end up with an integral which diverges logarithmically in the limit $T_c \rightarrow 0$. Use integration by parts to extract the divergent part. In order to solve the remaining (convergent) integral, you may use the limit $T_c \rightarrow 0$. The remaining integral will be given by

$$\int_0^\infty dx \frac{\ln x}{\cosh^2(x)} = -\gamma + \ln\left(\frac{\pi}{4}\right) ,$$

with $\gamma = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{k} - \ln n \right] \approx 0.5772$ being the Euler-Mascheroni constant.

- d) *Bonus:* Compute the integral given above.

Hint: You may use that

$$\ln x = \lim_{n \rightarrow 0} \frac{x^n - 1}{n} ,$$

and you may change the order of integration and applying the limit $n \rightarrow 0$. In case your favourite computer algebra program fails to solve the integrals, consider looking them up in a table of integrals like the book by Gradshteyn and Ryzhik.