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Advanced Statistical Physics - Problem Set 13

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Due Date: Tuesday, July 14, 10:00 a.m., solutions must be mailed to stp2.leipziguni@gmail.com

Internet: Advanced Statistical Physics exercises

1. Field operators

3 Points

The operators $a_k^\dagger$ and a_k create or annihilate single particle states with momentum k, respectively. They obey the commutation relations $[a_k, a_{k'}]_\zeta = 0$, and $[a_k, a_{k'}^\dagger]_\zeta = \delta_{k,k'}$ with $\zeta = 1$ for bosons and $\zeta = -1$ for fermions. The field operator $\Psi(x)$ is defined as the Fourier transform

$$\Psi(x) = \frac{1}{\sqrt{L}} \sum_k a_k e^{ikx} \ .$$

Show that $\Psi(x)$ and its Hermitian conjugate $\Psi^\dagger(x)$ obey the commutation relations

$$[\Psi(x), \Psi^\dagger(y)]_\zeta = \delta(x-y) \ .$$

2. Lindhard response function

2 + 3 + 1 + 3 + 2 Points

As derived in the lectures, the response of a d-dimensional free electron gas to a potential $\phi(\boldsymbol{r})$ is described by an induced charge $\varrho^{\text{ind}}(\boldsymbol{r}, t)$ with Fourier transform

$$\varrho^{\text{ind}}(\boldsymbol{q}, \omega) = \Pi_0(\boldsymbol{q}, \omega)\, \phi(\boldsymbol{q}, \omega) \ ,$$

where $\Pi_0(\boldsymbol{q}, \omega)$ is the Lindhard response function given by

$$\Pi_0(\boldsymbol{q}, \omega) = 2 \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \, \frac{f_{\boldsymbol{k}} - f_{\boldsymbol{k}+\boldsymbol{q}}}{\xi_{\boldsymbol{k}} - \xi_{\boldsymbol{k}+\boldsymbol{q}} + i\omega} \ .$$

The aim of this task is to compute the Lindhard response function for $T = 0$ up to linear order in $\boldsymbol{q}$.

(a) Show that the Fermi function

$$f_{\boldsymbol{k}} = f(\xi_{\boldsymbol{k}}, T) = \frac{1}{1 + \exp\left[\frac{\xi_{\boldsymbol{k}}}{k_B T}\right]}$$

reduces to the a Heaviside step function $\theta(-\xi_{\boldsymbol{k}})$ in the limit $T \to 0^+$. Note that $\xi_{k=k_F} = 0$.

(b) Expand $f_{\boldsymbol{k}} - f_{\boldsymbol{k}+\boldsymbol{q}}$ and $\xi_{\boldsymbol{k}} - \xi_{\boldsymbol{k}+\boldsymbol{q}}$ up to linear order in $\boldsymbol{q}$ and show that

$$\Pi_0(\boldsymbol{q}, \omega) \approx 2 \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \, \delta(\xi_{\boldsymbol{k}}) \frac{\frac{\hbar^2}{m} \boldsymbol{k} \cdot \boldsymbol{q}}{-\frac{\hbar^2}{m} \boldsymbol{k} \cdot \boldsymbol{q} + i\omega} \ .$$

(c) Argue that you can use $\boldsymbol{k} = k_F \boldsymbol{n}$ in the integral, with $\boldsymbol{n}$ denoting the direction of $\boldsymbol{k}$. Show that $\Pi_0(\boldsymbol{q}, \omega)$ is given by

$$\Pi_0(\boldsymbol{q}, \omega) = 2 \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \, \delta(\xi_{\boldsymbol{k}}) \frac{\hbar v_F \boldsymbol{n} \cdot \boldsymbol{q}}{-\hbar v_F \boldsymbol{n} \cdot \boldsymbol{q} + i\omega} \ ,$$

with $v_F = \hbar k_F / m$ denoting the Fermi velocity.

(d) Show that for $d = 3$ the Lindhard function is given by

$$\Pi_0(\boldsymbol{q}, \omega) = -\rho_F \left[1 + \frac{i\omega}{2\hbar v_F q} \ln \left(\frac{i\omega - \hbar v_F q}{i\omega + \hbar v_F q} \right) \right] \ ,$$

with the Fermi density of states given by

$$\rho_F = 2 \int \frac{d^d k}{(2\pi)^d} \delta(\xi_{\boldsymbol{k}}) \ .$$

(e) Calculate the static limit $\Pi_0(\boldsymbol{q}, \omega \to 0)$ for the case $d = 3$. Does the result depend on the dimension of system?