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Advanced Statistical Physics - Problem Set 11

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Due Date: Tuesday, June 30, 10:00 a.m., solutions must be mailed to stp2.leipziguni@gmail.com

1. KosterlitzThouless transition

4+2+3 Points

The KosterlitzThouless transition is a second order phase transition in the two-dimensional xy-model due to topological defects. We describe $\theta(\boldsymbol{x})$ as the spatial orientation of a spin-density $\boldsymbol{m}(\boldsymbol{x}) = \overline{m}\,(\cos\theta(\boldsymbol{x})\boldsymbol{e}_x + \sin\theta(\boldsymbol{x})\boldsymbol{e}_y)$. We require $\boldsymbol{m}(\boldsymbol{x})$ to be continuous everywhere except at a single point and consider a vortex with

$$\theta(r,\varphi) = n\varphi + \theta_0$$

where r and φ are polar coordinates and n is an integer. The singularity at the origin is removed by requiring $\langle \boldsymbol{m}(\boldsymbol{x})\rangle = 0$ inside the vortex core of size a, centered around the origin.

a) Using the definition of $\theta(r,\varphi)$, explicitly calculate the integral

$$\frac{1}{2\pi}\int_{\mathcal{C}} \mathrm{d}\boldsymbol{l}\cdot\nabla\theta$$

where $\mathcal{C}$ is a circle with radius R around the origin. Give an interpretation for the meaning of the variable n. Comment on the behavior of the angle θ at the positive x-axis, i.e. on $\theta(x,0)$? How is this compatible with continuity of the order parameter $\boldsymbol{m}(\boldsymbol{x})$?

b) The energy of a vortex in a circular volume Ω with radius R is divided into two contributions. i) the energy E_{c} of the vortex core. The core is a circular region with radius a around the center of the vortex. ii) the elastic energy $E_{\mathrm{el}} = (K/2)\int \mathrm{d}^2x(\nabla\theta)^2$. Show that the elastic energy is given by

$$E_{\mathrm{el}} = \pi K n^2 \ln\left(\frac{R}{a}\right) \;.$$

c) Obtain an expression for the free energy $F = E_{\mathrm{el}} - TS$ of a vortex. Find the transition temperature T_c above which vortices can proliferate and destroy the ordered phase.

Hint: To obtain an expression for the entropy, you may want to estimate the number of different vortex positions in the area Ω by taking into account the finite area πa^2 of the vortex core. To find the transition temperature, consider at which temperature the contribution of the vortex to the free energy changes sign.

2. Hartree approximation

2+3+3+1 Points

Consider the Landau-Ginzburg Hamiltonian:

$$\beta\mathcal{H} = \int d^d x \left[\frac{t}{2}\boldsymbol{m}^2 + \frac{K}{2}(\nabla \boldsymbol{m})^2 + u(\boldsymbol{m}^2)^2 \right],$$

describing an N-component magnetization vector $\boldsymbol{m}(\boldsymbol{x})$. Assume that $t > 0$.

a) Perform a Hubbard-Stratonovich transformation by first multiplying the partition function by

$$\mathbb{1} = \int D\rho(\mathbf{x}) e^{-N^2 \int d^d x \rho(\mathbf{x})^2/2}$$

and performing a shift $\rho \to \rho + \alpha \boldsymbol{m}^2$ and show that with suitably chosen α you obtain a new Hamiltonian

$$\beta\mathcal{H}[m,\rho] = \int d^d x \left[\frac{t + 2N^2\alpha\rho}{2}\boldsymbol{m}^2 + \frac{K}{2}(\nabla \boldsymbol{m})^2 + \frac{N^2\rho^2}{2} \right] .$$

b) We want to find saddle-point equation, where $\rho(\mathbf{x}) = \rho_0$. Therefore, assume that ρ is constant in space and integrate over $\boldsymbol{m}$ so that you will obtain an effective Hamiltonian for ρ_0

$$\beta H_{\text{eff}}(\rho_0) = \frac{N^2\rho_0^2 V}{2} + \frac{N}{2}\sum_{\mathbf{q}} \ln(t + 2N^2\alpha\rho_0 + Kq^2) .$$

c) Use the effective Hamiltonian obtained in (b) to find the saddle-point equation for ρ_0. Notice that in the Hamiltonian obtained in part (a) t has been renormalized so that $t' = t + 2N^2\alpha\rho$. Use the saddle-point equation to find the self-consistency equation for t'

$$t' = t + \frac{4uN}{(2\pi)^d}\int d^d q \frac{1}{t' + Kq^2} .$$

d) Argue why the method used above works well in the limit $N \to \infty$.