Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow
N. John, A. Afanah

Advanced Statistical Physics - Problem Set 9

Summer Term 2020

Due Date: Tuesday, June 16, 10:00 a.m., solutions must be mailed to stp2.leipziguni@gmail.com

1. Longitudinal susceptibility

2+2+2+2+2+2+2 Points

This problem is intended to show you the origin of the divergence of the longitudinal susceptibility in dimensions $d < 4$. There are a number of subtleties in this calculation which you are instructed to ignore at various steps. You may want to think about why they are justified.
Consider the Landau-Ginzburg Hamiltonian:

$$\beta\mathcal{H} = \int d^d x \left[\frac{t}{2}\vec{m}^2 + \frac{K}{2}(\nabla\vec{m})^2 + u(\vec{m}^2)^2\right],$$

describing an n-component magnetization vector $\vec{m}$, in the ordered phase for $t < 0$.

(a) Let $\vec{m}(\mathbf{x}) = [\bar{m} + \phi_l(\mathbf{x})]\hat{e}_l + \vec{\phi}_t(\mathbf{x})$, and expand $\beta\mathcal{H}$ keeping all terms in the expansion.

(b) Regard the quadratic terms in ϕ_l, and $\vec{\phi}_t$ as an unperturbed Hamiltonian $\beta\mathcal{H}_0$, and the lowest order term coupling ϕ_l and $\vec{\phi}_t$ as a perturbation U; i.e.

$$U = 4u\bar{m}\int d^d x \phi_l \vec{\phi}_t^{\,2}$$

Write U in Fourier space in terms of $\phi_l(\mathbf{q})$ and $\vec{\phi}_t(\mathbf{q})$.

(c) Calculate the Gaussian (bare) expectation values $\langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\rangle_0$ and $\langle\phi_{t,\alpha}(\mathbf{q})\phi_{t,\beta}(\mathbf{q}')\rangle_0$, and the corresponding momentum dependent susceptibilities $\chi_l(\mathbf{q})_0$ and $\chi_t(\mathbf{q})_0$.

(d) Calculate $\langle\vec{\phi}_t(\mathbf{q}_1)\cdot\vec{\phi}_t(\mathbf{q}_2)\ \vec{\phi}_t(\mathbf{q'}_1)\cdot\vec{\phi}_t(\mathbf{q'}_2)\rangle_0$ using Wick's theorem (see the second problem below).

(e) Write down the expression for $\langle\phi_l(\mathbf{q})\phi_l(\mathbf{q}')\rangle$ to second-order in the perturbation U. Note that since U is odd in ϕ_l, only two terms at the second order are non-zero.

(f) Using the form of U in Fourier space, find an expression for the correction term obtained in part (e). Note that only the connected terms for the longitudinal correlation function need to be evaluated.

(g) Ignore the disconnected term [i.e. the part proportional to $(n-1)^2$], and write down an expression for $\chi_l(\mathbf{q})$ in the second order perturbation theory.

(h) Show that for $d < 4$, the correction term diverges as q^{d-4} for $q \to 0$, implying an infinite longitudinal susceptibility.

2. Wick's theorem

Bonus +4 Points

Assume a Gaussian Hamiltonian

$$\beta\mathcal{H} = \frac{1}{2}\sum_{i,j} K_{ij}\phi_i\phi_j.$$

(a) Calculate the expectation value $\langle\phi_{i_1}\phi_{i_2}\rangle$.

(b) Let $F[\phi] = \prod_{k=2}^{2n}\phi_{i_k}$. Show that

$$\langle\phi_{i_1}F[\phi]\rangle = \sum_j \langle\phi_{i_1}\phi_j\rangle\langle\frac{\partial}{\partial\phi_j}F[\phi]\rangle.$$

(c) Proof the Wick's theorem: Let $\mathcal{P}_{2n}$ be the set of all pairings of $2n$-elements. Then

$$\langle\prod_{k=1}^{2n}\phi_{i_k}\rangle = \sum_{P\in\mathcal{P}_{2n}}\prod_{(p_1,p_2)\in P}\langle\phi_{p_1}\phi_{p_2}\rangle.$$

For example this means that: $\langle\phi_1\phi_2\phi_3\phi_4\rangle = \langle\phi_1\phi_2\rangle\langle\phi_3\phi_4\rangle + \langle\phi_1\phi_3\rangle\langle\phi_2\phi_4\rangle + \langle\phi_1\phi_4\rangle\langle\phi_2\phi_3\rangle$.
Hint: Use induction and the result obtained in (b). Notice that all pairings of $2n$ elements can be realized by pairing of i_1 with each i_k ($k \geq 2$) and considering all possible pairings of the remaining $2n-2$ elements.