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Advanced Statistical Physics - Problem Set 11

Summer Term 2019

Due Date: Wednesday, June 26, 12:00 p.m., Hand in tasks marked with * to mailbox with label 'Advanced Statistical Physics Exercises' inside ITP room 105b

Internet: Advanced Statistical Physics exercises

16. Central limit theorem

2 + 2 + 3 Points

The aim of this task is to prove the central limit theorem using characteristic functions.

Consider a set $\{y_1, ..., y_N\}$ of independent random variables which all follow the same distribution function $p(y)$ with the same mean $\mu = \langle y_i \rangle$ and variance $\sigma^2 = \langle (y_i - \mu)^2 \rangle$. The sum of these random variables is denoted as

$$S_N = \sum_{y=1}^{N} y_i \ .$$

The central limit theorem states that $Z_N = (S_N - N\mu)/(\sigma\sqrt{N})$ converges to the standard normal distribution with mean 0 and standard deviation 1 in the limit $N \to \infty$.

a) Compute the mean and standard deviation of S_N.

b) Show that the characteristic function of Z_N is given by

$$\langle \mathrm{e}^{iqZ_N} \rangle = \left[\left\langle \mathrm{e}^{iq(y_1-\mu)/(\sqrt{N}\sigma)} \right\rangle \right]^N \ .$$

Hint: the distribution function of independent random variables is given by the product of the individual involved distribution functions.

c) Taylor expanding the characteristic function and computing the limit $N \to \infty$, show the central limit theorem, i.e.

$$\lim_{N\to\infty} \langle \mathrm{e}^{iqZ_N} \rangle = \mathrm{e}^{-\frac{1}{2}q^2} \ .$$

Argue why a Gaussian characteristic function with mean 0 and standard deviation 1 implies that the distribution function is also a Gaussian with the same cumulants.

17. Markowitz portfolio selection*

1 + 2 + 4 + 3 Points

This problem is a brief excursion into the world of economic physics using our knowledge of statistics.

In order to minimize the risk while having a good change of a high return (relative price change), it is advisable to invest in a portfolio of assets instead of a single risky asset. The goal is to minimize the risk σ for a fixed target return $\overline{r}$ of a portfolio.

We consider a portfolio at time $t = 0$ with wealth $W(0)$ consisting of N risky assets with values $S_i(0)$ and a risk-free asset with value $S_0(0)$. Here, n_i is the number of asset i in the portfolio. The wealth is therefore defined as $W(t) = \sum_i n_i S_i(t)$. To compute the return (at $t = \Delta t$), we define relative weights

$$q_i = \frac{n_i S_i(0)}{W(0)} \ .$$

We consider the idealized case of Gaussian assets ($S_i(\Delta T) - S_0(0)$ follows a normal distribution) and allow the different assets (S_i, S_j) to be correlated. We denote the return and standard deviation of the individual assets by $r_i = \langle \frac{S_i(0)-S_i(\Delta T)}{S_i(0)} \rangle$ and σ_i, respectively, where the portfolio return r is defined as

$$r = \langle \frac{W(0) - W(\Delta T)}{W(0)} \rangle \ .$$

a) Show that the return r is given by

$$r = \sum_{i=0}^{N} q_i r_i \ .$$

b) Show that the variance (risk) $\langle (r - \langle r \rangle)^2 \rangle$ is given by

$$\langle (r - \langle r \rangle)^2 \rangle = \sum_{i,j=1}^{N} q_i q_j \sigma_i \sigma_j C_{ij}$$

where $C_{ij} = (\langle r_i r_j \rangle - \langle r_i \rangle \langle r_j \rangle)/(\sigma_i \sigma_j)$ is the matrix of correlations between the assets.

c) Fix an expected return $r = \overline{r}$ and find an optimal set of weights q_i^* that minimize the risk from (b).

Hints: In order to include the constraint of fixed r you might want to chose a Lagrange multiplier λ and use that $q_0 = 1 - \sum_{i=1}^{N} q_i$.

d) Determine the minimal portfolio variance (risk) $\sigma_W^2 = \langle (r - \langle r \rangle)^2 \rangle \big|_{q_i = q_i^*}$ for return $r = \overline{r}$ and show that $\sigma_W^2 \propto (\overline{r} - r_0)^2$. This means that a higher possible return is always accompanied with a higher risk $\overline{r} - r_0 \propto \sigma_W$.