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Advanced Statistical Physics - Problem Set 9

Summer Term 2019

Due Date: Wednesday, June 12, 12:00 p.m., Hand in tasks marked with * to mailbox with label 'Advanced Statistical Physics Exercises' inside ITP room 105b

Internet: Advanced Statistical Physics exercises

12. Cooper pair size

2+4+1 Points

a) The Cooper pair wave function can be expanded in a plane-wave basis as

$$\psi(\boldsymbol{r}) = \sum_{\boldsymbol{k}} g(\boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{r}} \ ,$$

with $g(\boldsymbol{k})$ being the amplitude of finding an electron in state with momentum $\boldsymbol{k}$ and another one in a state with momentum $-\boldsymbol{k}$, and $\boldsymbol{r}$ being the relative coordinate (distance between the electrons) of the Cooper pair. Note that $g(\boldsymbol{k}) = 0$ for $k < k_F$. Start with the definition of the mean square radius

$$R^2 = \frac{\int d\boldsymbol{r}\, r^2 |\psi(\boldsymbol{r})|^2}{\int d\boldsymbol{r}\, |\psi(\boldsymbol{r})|^2} \ ,$$

and show that it can be reexpressed as

$$R^2 = \frac{\sum_{\boldsymbol{k}} |\nabla_{\boldsymbol{k}}\, g(\boldsymbol{k})|^2}{\sum_{\boldsymbol{k}} |g(\boldsymbol{k})|^2} \ .$$

b) By turning the sums into integrals and by using the definition of the density of state, show that the expression for the mean square radius becomes

$$R^2 \simeq \frac{\left(\frac{\partial \xi}{\partial k}\right)^2_{\xi=0} \int\limits_0^\infty d\xi \left(\frac{\partial g(\xi)}{\partial \xi}\right)^2}{\int\limits_0^\infty d\xi\, g(\xi)^2} \ .$$

Further, use that $g(\xi) \propto 1/(\Delta + 2\xi)$ to show that the mean square radius of a Cooper pair is given by

$$R = \frac{2}{\sqrt{3}} \frac{\hbar v_F}{\Delta} \ ,$$

with v_F being the Fermi velocity, and Δ being the binding energy of the Cooper pair relative to the Fermi surface.

c) Insert realistic values for v_F and Δ and estimate the size of the Cooper pair.

13. Correlation function II*

3+3+3 Points

Consider the Ginzburg-Landau functional

$$\mathcal{H} = \int d^d x \left[\frac{a\tau}{2}\psi(\mathbf{x})^2 + \frac{c}{2}(\nabla\psi(\mathbf{x}))^2 - h(\mathbf{x})\psi(\mathbf{x}) \right] .$$

The associated Euler-Lagrange equation is given by

$$c\nabla^2\psi(\mathbf{x}) = a\tau\psi(\mathbf{x}) - h(\mathbf{x}) .$$

a) Use the Fourier transformation to write down the formal solution of this equation for $h(\mathbf{x}) = h\delta^{(d)}(\mathbf{x})$. In the lectures it will be shown that this solution is equivalent with a two point correlation function.

b) Solve the Euler-Lagrange equation for $\tau = 0$ and $h(\mathbf{x}) = h\delta^{(d)}(\mathbf{x})$.
Hint: Use Gauss's theorem.

c) Solve the Euler-Lagrange equation for $\tau > 0$.
Hint: Assume that the solution is spherically symmetric and decays exponentially at large distances

$$\psi(\mathbf{x}) \propto \frac{e^{-r/\xi}}{r^p}.$$

Solve the equation in the limits $r \ll \xi$ and $r \gg \xi$.