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Advanced Statistical Physics - Problem Set 2

Summer Term 2019

Due Date: Wednesday, April 17, 12:00 p.m., Hand in tasks marked with * to mailbox inside ITP room 105b

Internet: Advanced Statistical Physics exercises

2. Field operators *

3 Points

The operators $a_k^\dagger$ and a_k create or annihilate single particle states with momentum k, respectively. They obey the commutation relations $[a_k, a_{k'}]_\zeta = 0$, and $[a_k, a_{k'}^\dagger]_\zeta = \delta_{k,k'}$ with $\zeta = 1$ for bosons and $\zeta = -1$ for fermions. The field operator $\Psi(x)$ is defined as the Fourier transform

$$\Psi(x) = \frac{1}{\sqrt{L}} \sum_k a_k e^{ikx} \ .$$

Show that $\Psi(x)$ and its Hermitian conjugate $\Psi^\dagger(x)$ obey the commutation relations

$$[\Psi(x), \Psi^\dagger(y)]_\zeta = \delta(x - y) \ .$$

3. Second quantization with field operators

4+5+1 Points

A many-particle state is described by the Hamiltonian

$$H = \sum_{j=1}^{N} \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_j^2} + U(x_j) \right) + \frac{1}{2} \sum_{i \neq j} V(x_i - x_j) \ ,$$

with $U(x)$ being a scalar potential, $V(x-x')$ being the two-particle interaction potential, and the wave function $\varphi_\alpha(x_1, x_2, \ldots, x_N)$ being an eigenstate of H with eigenvalue E_α. In the following, we denote the non-interacting part of the Hamiltonian by H_0 and the interaction part by H_{int}. In second quantized notation, the wave function φ_α is described by a state vector

$$|\varphi_\alpha\rangle = \int dx_1 dx_2 \ldots dx_N \, \varphi_\alpha(x_1, x_2, \ldots, x_N) \Psi^\dagger(x_1) \Psi^\dagger(x_2) \ldots \Psi^\dagger(x_N) |0\rangle \ ,$$

with commutation relations of Ψ and $\Psi^\dagger$ as in exercise 2, supplemented by the condition that the operator $\Psi(x)$ annihilates the vacuum state, $\Psi(x)|0\rangle = 0$. In second quantization, the Hamiltonian reads

$$H_s = \int dx \, \Psi^\dagger(x) \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x) \right) \Psi(x) + \frac{1}{2} \int dx \, dy \, \Psi^\dagger(x) \Psi^\dagger(y) V(x - y) \Psi(y) \Psi(x) \ .$$

As before, we denote the non-interacting part by $H_{0,s}$ and the interaction part by $H_{\text{int},s}$. The aim of the task is to show that this is a consistent notation by proving

$$H_s |\varphi_\alpha\rangle = E_\alpha |\varphi_\alpha\rangle \ .$$

(a)* Start with the non-interacting part $H_{0,s}$ and show that

$$H_{0,s}|\varphi_\alpha\rangle = \int dx_1 dx_2 \dots dx_N \, [H_0 \varphi_\alpha(x_1, x_2, \dots, x_N)] \Psi^\dagger(x_1)\Psi^\dagger(x_2)\dots\Psi^\dagger(x_N)|0\rangle \ .$$

Hint: Use the result from exercise 2, i.e.

$$[\Psi(x), \Psi^\dagger(y)]_\zeta = \delta(x-y) \ ,$$

with $\zeta = \pm 1$ to cover bosons and fermions simultaneously.

(b) For the interaction part $H_{\text{int},s}$ show that

$$H_{\text{int},s}|\varphi_\alpha\rangle = \int dx_1 dx_2 \dots dx_N \, [H_{\text{int}} \varphi_\alpha(x_1, x_2, \dots, x_N)] \Psi^\dagger(x_1)\Psi^\dagger(x_2)\dots\Psi^\dagger(x_N)|0\rangle \ .$$

(c) From your results of task (a) and (b) conclude that

$$H_s|\varphi_\alpha\rangle = E_\alpha|\varphi_\alpha\rangle \ .$$