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Statistical Mechanics of Deep Learning - Problem set 9

Winter Term 2024/25

The problem set will be discussed in the seminar on **Monday 16.12.2024, 9:15**.

18. Generalization of the maximally stable vector *4+4+4 Points*

To compute the generalization error of the maximally stable vector e.g. from the adatron rule, consider the corresponding cost function

$$V(\Delta) = \frac{1}{2}(\Delta - \kappa)^2\, \theta(\kappa - \Delta)$$

(a) Start by showing that $\Delta_0(x,t)$ which minimizes $V(\Delta) + \frac{(\Delta - t)^2}{2x}$, is given by

$$\Delta_0(x,t) = \begin{cases} \dfrac{t + \kappa x}{1+x} & , \ t < \kappa \\ t & , \ t \geq \kappa. \end{cases}$$

(b) Use the result obtained in part (a) in

$$1 - R^2 = 2\alpha \int \boldsymbol{D}t\ (\Delta_0(x,t) - t)^2\ H\left(-\frac{Rt}{\sqrt{1-R^2}}\right)$$

and

$$R = \frac{2\alpha}{\sqrt{2\pi(1-R^2)}} \int \boldsymbol{D}t\ \Delta_0(x,t) \exp\left(-\frac{R^2 t^2}{2(1-R^2)}\right)$$

and consider the limit $x \to \infty$ which corresponds to choosing the optimal κ. Show that the above equations reduces to

(1) $$1 - R^2 = 2\alpha \int_{-\infty}^{\kappa_{opt}} \boldsymbol{D}t\ (\kappa_{opt} - t)^2\ H\left(-\frac{Rt}{\sqrt{1-R^2}}\right)$$

and

(2) $$R = \frac{2\alpha}{\sqrt{2\pi(1-R^2)}} \int_{-\infty}^{\kappa_{opt}} \boldsymbol{D}t\ (\kappa_{opt} - t) \exp\left(-\frac{R^2 t^2}{2(1-R^2)}\right)$$

(c) Equations (1) and (2) can be used to determine $\kappa_{opt}(\alpha)$ and $R(\alpha)$ numerically. From the latter the generalization error can be computed using

$$\varepsilon\ = \frac{1}{\pi}\arccos(R).$$

Show that in the limit $\alpha \to \infty$, the asymptotic behavior of the generalization error is

$$\varepsilon(\alpha) \sim \frac{c}{\alpha \int_{-\infty}^{1} du (1-u) e^{-u^2/2c}} \sim \frac{0.5005}{\alpha}$$

with the constant c determined by the transcendental equation

$$\sqrt{\frac{c}{2\pi}} = \frac{\int_{-\infty}^{1} du (1-u)^2 H(-u/\sqrt{c})}{\int_{-\infty}^{1} du (1-u) e^{-u^2/2c}}$$