Institut für Theoretische Physik
Universität Leipzig

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Advanced Quantum Mechanics - Bonus Problem Set

Winter Term 2019/20

Due Date: Only if you are below 50% of the total points, hand in solutions to problems marked with * before the lecture on **Friday, 07.02.2020, 09:15**.

This exercise sheet is **not mandatory**, but you can solve it to get additional points. In case that you already have at least 50% of the points from the exercises, it will not be marked. You need a total of at least **121.5 points** to be admitted to the exam.

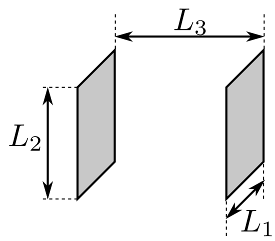
The **exam** will take place on **February 26 at 10:00 a.m. in the Theoretical Lecture Hall**. (Please also check the official website from the faculty in case that there are any updates: https://www.physgeo.uni-leipzig.de/en/study/exams/)

*36. Casimir Effect

4+4+2+1 Points

As shown in problem 28, the Hamiltonian of the quantized radiation field confined to a box with volume $V = L_1 L_2 L_3$ and with periodic boundary conditions, is given by

$$H = \sum_{\boldsymbol{k}} \sum_{\lambda=\pm} \hbar\omega_{\boldsymbol{k}} \left(a^\dagger_{\boldsymbol{k},\lambda} a_{\boldsymbol{k},\lambda} + \frac{1}{2} \right) , \quad \omega_{\boldsymbol{k}} = c|\boldsymbol{k}| , \quad k_i = \frac{\pi}{L_i} n_i , \quad n_i \in \mathbb{N} .$$



In particular we found that the ground state, in which no modes are excited, has a divergent energy. Whilst this divergent vacuum zero-point energy is not observable, the dependence on the boundaries does lead to observable phenomena.
To investigate this, we consider in the following two conducting plates with surface areas $A = L_1 L_2$ seperated by a distance L_3. In the plane of the plates we will still be using periodic boundary conditions and consider the limit $L_1, L_2 \to \infty$. Since the electric field $\boldsymbol{E}$ on the plates vanishes, only modes with $|\boldsymbol{E}| \propto \sin(k_3 x_3)$ are possible. Here $k_3 = n_3\pi/L_3$ with $n_3 = 1, 2, \ldots$. To get a finite vacuum energy we will moreover introduce an exponential cutoff $e^{-\epsilon\omega_{\boldsymbol{k}}}$ with $\epsilon > 0$, and take the limit of $\epsilon \to 0$ at the end of the calculation. The energy density per unit plate area between the plates is given by

$$\begin{aligned}\sigma_E(L_3) &= \lim_{L_1, L_2 \to \infty} \frac{1}{L_1 L_2} \sum_{\boldsymbol{k}} \hbar\omega_{\boldsymbol{k}}\, e^{-\epsilon\omega_{\boldsymbol{k}}} \\ &= \hbar c \sum_{n_3=1}^{\infty} \int \frac{d^2k}{(2\pi)^2} \sqrt{k_1^2 + k_2^2 + (\tfrac{\pi n_3}{L_3})^2}\, e^{-\epsilon c\sqrt{k_1^2+k_2^2+(\frac{\pi n_3}{L_3})^2}}\end{aligned}$$

(a) Using polar coordinates and a suitable subsitution show that $\sigma_E(L_3)$ can be written as

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2} \frac{\partial^2}{\partial\epsilon^2} \sum_{n=1}^{\infty} \int_{n\pi c/L_3}^{\infty} d\omega e^{-\epsilon\omega} .$$

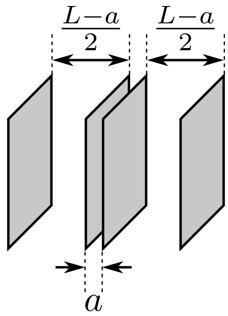
(b) Calculate the integral over ω and perform the sum to show that

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2}\frac{\partial^2}{\partial\epsilon^2}\left(\frac{1}{\epsilon}\frac{1}{e^{\epsilon\pi c/L_3}-1}\right).$$

Show further that

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2}\left(\frac{6}{\epsilon^4}\frac{L_3}{\pi c} - \frac{1}{\epsilon^3} - \frac{1}{360}\left(\frac{\pi c}{L_3}\right)^3 + \mathcal{O}(\epsilon^2)\right).$$

(c) The energy density calculated in the previous part diverges as the distance between the plates increases ($L_3 \to \infty$). This will be our reference point. We therefore consider two plates separated by a fixed distance a, together with two external plates which are places a further distance $(L-a)/2$ away. The relevant energy density is then given by



$$\sigma_E(a, L) = \sigma_E(a) + 2\sigma_E\left(\frac{L-a}{2}\right).$$

Find an expression for $\sigma_E(a, L)$ using your result in (b).

(d) Since the energy density varies with the distance between plates, the plates experience a pressure which is given by

$$p_{\text{vac}} = -\lim_{L\to\infty}\frac{\partial}{\partial a}\sigma_E(a, L).$$

How large is this pressure for $A = 1\,\text{cm}^2$ and $a = 1\,\mu\text{m}$?

*37. Zitterbewegung

2+2+2 Points

In this problem we will consider the Dirac Hamiltonian

$$\hat{H}_D = c\boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta mc^2,$$

where m is the mass of the particle, c is the speed of light, and $\boldsymbol{\alpha}$ and β are matrices given by

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

with $\boldsymbol{\sigma}$ denoting the vector of Pauli matrices and I_2 denoting the 2×2 unit matrix. The Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Show that the velocity operator is given by $\hat{\boldsymbol{v}} = c\boldsymbol{\alpha}$.

Hint: You may use the Heisenberg equation of motion which states that an operator $\hat{A}$ which does not explicitly depend on time satisfies $-i\hbar\dot{\hat{A}} = [\hat{H}, \hat{A}]$.

(b) Consider now a Dirac particle at rest in a volume V. A general eigenspinor can then be written as

$$\psi = \frac{1}{\sqrt{2V}} \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imc^2t/\hbar} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{imc^2t/\hbar} \right].$$

Give a physical interpretation of the two terms in the spinor.

(c) Derive an expression for $\langle \hat{v}_z \rangle = \langle \psi | \hat{v}_z | \psi \rangle$ using the spinor defined in the previous part of the problem. Comment on your result.