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Advanced Quantum Mechanics - Problem Set 8

Winter Term 2019/20

Due Date: Hand in solutions to problems marked with * before the lecture on **Friday, 13.12.2019, 09:15**. The problem set will be discussed in the tutorials on Wednesday, 18.12.2019, and Friday, 20.12.2019.

*20. Commutators of Dirac matrices *2+2 Points*

Consider the Dirac matrices

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix},$$

where $\boldsymbol{\sigma}$ is the vector of Pauli matrices and $\mathbb{1}_2$ is the 2-dimensional unit matrix. Define also

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}.$$

Show that (i) $\beta\Sigma_i = \Sigma_i\beta$, and that (ii) $[\alpha_i, \Sigma_j] = 2i\epsilon_{ijk}\alpha_k$.

21. Four-current for the free particle solutions of the Dirac equation *1+4 Points*

The free particle solutions of the Dirac equation can be written using

$$\boldsymbol{u}_R^{(+)}(p) = \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E_p+m} \\ 0 \end{pmatrix}, \qquad \boldsymbol{u}_L^{(+)}(p) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{E_p+m} \end{pmatrix},$$

for solutions with positive energy $E = E_p$, and

$$\boldsymbol{u}_R^{(-)}(p) = \begin{pmatrix} \frac{-p}{E_p+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \boldsymbol{u}_L^{(-)}(p) = \begin{pmatrix} 0 \\ \frac{p}{E_p+m} \\ 0 \\ 1 \end{pmatrix},$$

for solutions with negative energy $E = -E_p$.

(a) What are the free-particle wave-functions?

(b) Calculate the four-current $j^\mu = \bar{\Psi}\gamma^\mu\Psi$, where $\bar{\Psi} = \Psi^\dagger\beta$. Interpret your result.

22. Klein-Gordon equation in an electromagnetic field *2 Points*

The Klein-Gordon equation is given by

$$\left(\partial_\mu \partial^\mu + m^2\right) \Psi(\boldsymbol{x}, t) = 0.$$

Show that, in an electromagnetic field with four-potential $A^\mu = (\Phi, \boldsymbol{A})$, the Klein-Gordon equation becomes

$$\left(D_\mu D^\mu + m^2\right) \Psi(\boldsymbol{x}, t) = 0,$$

with $D_\mu = \partial_\mu + ieA_\mu$.

*23. Chiral Symmetry *1+1+1+1+2+3 Points*

Define the fifth γ-matrix as $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and consider the Dirac Hamiltonian

$$H_D = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m,$$

with

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}.$$

(a) Show that $\{\gamma^\mu \partial_\mu, \gamma^5\} = 0$. The first term in the anti-commutator is known as the Dirac operator. Since the Dirac Hamiltonian can be constructed using $\gamma^0\gamma^i = \alpha^i$ and $\gamma^0 = \beta$, the Hamiltonian anti-commutes with the operator $i\gamma^1\gamma^2\gamma^3$.

(b) Consider now an operator $\hat{C}$ with the property that $\hat{C}^2 = \mathbb{1}$ and $\{\hat{H}, \hat{C}\} = 0$. Show that if $|E_n\rangle$ is an eigenstate of the Hamiltonian H with eigenvalue E_n, then $|-E_n\rangle = C|E_n\rangle$ is also an eigenstate of the Hamiltonian with eigenvalue $-E_n$.

(c) Using that $\sigma^l\sigma^m = i\varepsilon_{lmk}\sigma^k + \delta_{lm}\mathbb{1}_2$ show that $\gamma^l\gamma^m = -i\varepsilon_{lmk}\Sigma^k - \delta_{lm}\mathbb{1}_4$ for $l, m = 1, 2, 3$. Here $\Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$.

(d) Show that $\left(\gamma^5\right)^2 = \mathbb{1}_4$.

(e) Consider now a two-level system with energy eigenvalues $\pm E_n$. Write down the matrix representations of $\hat{C}$ and $\hat{H}$, and show that H is off-diagonal in the basis where C is diagonal.

(f) Generalize your result in (e) to N non-degenerate levels. That is show that it is possible to diagonalize C in such a way that H becomes off-diagonal. What happens qualitatively when there are degenerate eigenstates?

Hint: Diagonalize C (you know its eigenvalues!). You can construct H using your result in (e). Write down a suitable basis. Think about how to rearrange the rows and columns of your matrices such that the diagonal elements in C are sorted with the positive eigenvalues coming before the negative eigenvalues.