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Quantum Mechanics 2 - Problem Set 10

Wintersemester 2017/2018

Abgabe: The problem set will be discussed in the tutorials on **Thursday, 04.01.2018, 11:00** (German) and **Friday, 05.01.2018, 13:30** (English).

31. Singlet and triplet states

4+3+1 Punkte

Consider two angular momenta $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_2$ with $s_1 = s_2 = 1/2$. In this problem we will calculate the eigenvalues and eigenstates of $\hat{\mathbf{S}}^2$, where $\hat{\mathbf{S}} = \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2$. The eigenstates can be written as linear combinations of the 4 basis states

$$|s_1 = 1/2, s_2 = 1/2; m_1, m_2\rangle, \qquad \text{with } m_1, m_2 = -1/2, 1/2.$$

(a) Construct the 4×4 matrix representation of the operator $\hat{\mathbf{S}}^2$ in this basis.

(b) Calculate the eigenvalues of $\hat{\mathbf{S}}^2$ by diagonalising the matrix.

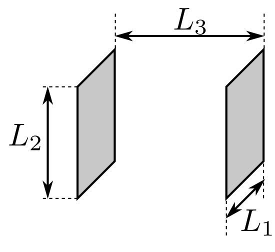
(c) Calculate the corresponding eigenstates.

32. Casimir Effect

4+4+3+1 Punkte

As shown in problem 30, the Hamiltonian of the quantised radiation field confined to a box with volume $V = L_1 L_2 L_3$ and with periodic boundary conditions, is given by

$$H = \sum_{\mathbf{k}} \sum_{\lambda=\pm} \hbar\omega_{\mathbf{k}} \big(a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} + \frac{1}{2}\big) \ , \quad \omega_{\mathbf{k}} = c|\mathbf{k}| \ , \quad k_i = \frac{2\pi}{L_i} n_i \ , \quad n_i \in \mathbb{N} \ .$$



In particular we found that the ground state, in which no modes are excited, has a divergent energy. Whilst this divergent vacuum zero-point energy is not observable, the dependence on the boundaries does lead to observable phenomena.
To investigate this, we consider in the following two conducting plates with surface areas $A = L_1 L_2$ seperated by a distance L_3. In the plane of the plates we will still be using periodic boundary conditions and consider the limit $L_1, L_2 \to \infty$. Since the electric field $\mathbf{E}$ between the plates vanishes, only modes with $|\mathbf{E}| \propto \sin(k_3 x_3)$ are possible. Here $k_3 = n_3\pi/L_3$ with $n_3 = 1, 2, \ldots$. To get a finite vacuum energy we will moreover introduce an exponential cutoff $e^{-\epsilon\omega_{\mathbf{k}}}$ with $\epsilon > 0$, and take the limit of $\epsilon \to 0$ at the end of the calculation. The energy density per unit plate area between the plates is given by

$$
\begin{aligned}
\sigma_E(L_3) &= \lim_{L_1,L_2\to\infty} \frac{1}{L_1 L_2} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}\, e^{-\epsilon\omega_{\mathbf{k}}} \\
&= \hbar c \sum_{n_3=1}^{\infty} \int \frac{d^2k}{(2\pi)^2} \sqrt{k_1^2 + k_2^2 + (\tfrac{\pi n_3}{L_3})^2}\, e^{-\epsilon c\sqrt{k_1^2+k_2^2+(\frac{\pi n_3}{L_3})^2}}
\end{aligned}
$$

(a) Using polar coordinates and a suitable subsitution show that $\sigma_E(L_3)$ can be written as

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2} \frac{\partial^2}{\partial \epsilon^2} \sum_{n=1}^{\infty} \int_{n\pi c/L_3}^{\infty} d\omega e^{-\epsilon\omega} .$$

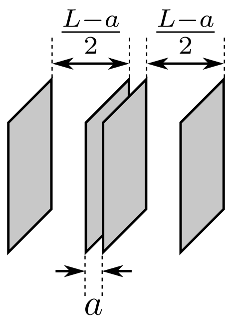
(b) Calculate the integral over ω and perform the sum to show that

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2} \frac{\partial^2}{\partial \epsilon^2} \Big(\frac{1}{\epsilon} \frac{1}{e^{\epsilon\pi c/L_3} - 1} \Big) .$$

Show further that

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2} \Big(\frac{6}{\epsilon^4} \frac{L_3}{\pi c} - \frac{1}{\epsilon^3} - \frac{1}{360} \Big(\frac{\pi c}{L_3}\Big)^3 + \mathcal{O}(\epsilon^2) \Big).$$

(c) The energy density calculated in the previous part diverges as the distance between the plates increases ($L_3 \to \infty$). This will be our reference point. We therefore consider two plates separated by a fixed distance a, together with two external plates which are places a further distance $(L-a)/2$ away. The relevant energy density is then given by



$$\sigma_E(a, L) = \sigma_E(a) + 2\sigma_E \left(\frac{L-a}{2} \right) .$$

Find an expression for $\sigma_E(a, L)$ using your result in (b).

(d) Since the energy density varies with the distance between plates, the plates experience a pressure which is given by

$$p_{\text{vac}} = - \lim_{L\to\infty} \frac{\partial}{\partial a} \sigma_E(a, L).$$

How large is this pressure for $A = 1\,\text{cm}^2$ and $a = 1\,\mu\text{m}$?

33. Bonus Problem *+5 Extra Punkte*

Consider N particles with angular momenta $l_1 = l_2 = ... = l_N = (N-1)/2$. Write down a state with total angular momentum $L = 0$. Explain why the total angular momentum for your state is zero. Is this state unique for $N = 1, 2, 3, 4$?
Hint: Think about the solutions of problems 29 and 31.