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Quantum Mechanics 2- Problem Set 7

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Abgabe: The problem set will be discussed in the tutorials on **Thursday, 30.11.2017, 11:00** (German) and **Friday, 01.12.2017, 13:30** (English).

21. Commutators of Dirac matrices

2+2 Punkte

Consider the Dirac matrices

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

where σ is the vector of Pauli matrices and I_2 is the 2-dimensional unit matrix. Define also

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}.$$

Show that (i) $\beta\Sigma_i = \Sigma_i\beta$, and that (ii) $[\alpha_i, \Sigma_j] = 2i\epsilon_{ijk}\alpha_k$.

22. Four-current for the free particle solutions of the Dirac equation

1+4 Punkte

The free particle solutions of the Dirac equation can be written using

$$u_R^{(+)}(p) = \begin{pmatrix} 1 \\ 0 \\ \frac{p}{E_p+m} \\ 0 \end{pmatrix}, \qquad u_L^{(+)}(p) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-p}{E_p+m} \end{pmatrix},$$

for solutions with positive energy $E = E_p$, and

$$u_R^{(-)}(p) = \begin{pmatrix} \frac{-p}{E_p+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad u_L^{(-)}(p) = \begin{pmatrix} 0 \\ \frac{p}{E_p+m} \\ 0 \\ 1 \end{pmatrix},$$

for solutions with negative energy $E = -E_p$.

(a) What are the free-particle wave-functions?

(b) Calculate the four-current $j^\mu = \bar{\Psi}\gamma^\mu\Psi$, where $\bar{\Psi} = \Psi^\dagger\beta$. Interpret your result.

23. Klein-Gordon equation in an electromagnetic field *2 Punkte*

The Klein-Gordon equation is given by

$$\left(\partial_\mu \partial^\mu + m^2\right) \Psi(\mathbf{x}, t) = 0.$$

Show that, in an electromagnetic field with four-potential $A^\mu = (\Phi, \mathbf{A})$, the Klein-Gordon equation becomes

$$\left(D_\mu D^\mu + m^2\right) \Psi(\mathbf{x}, t) = 0,$$

with $D_\mu = \partial_\mu + ieA_\mu$.

24. Chiral Symmetry *2+1+1+1+2+2 Punkte*

Define the fifth γ-matrix as $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and consider the Dirac Hamiltonian

$$H_D = \alpha \cdot \mathbf{p} + \beta m,$$

with

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

(a) Show that $\{\gamma^\mu \partial_\mu, \gamma^5\} = 0$. The first term in the anti-commutator is known as the Dirac operator. Since the Dirac Hamiltonian can be constructed using γ^0 and this term, the Hamiltonian anti-commutes with the operator $i\gamma^1\gamma^2\gamma^3$.

(b) Consider now an operator C with the property that $C^2 = 1$ and $\{H, C\} = 0$. Show that if $|E_n\rangle$ is an eigenstate of the Hamiltonian H with eigenvalue E_n, then $|-E_n\rangle = C|E_n\rangle$ is also an eigenstate of the Hamiltonian with eigenvalue $-E_n$.

(c) Calculate $\gamma^1\gamma^3$.

(d) Show that $\left(\gamma^5\right)^2 = 1$.

(e) Consider now a two-level system with energy eigenvalues $\pm E_n$. Write down the matrix representations of C and H, and show that H is off-diagonal in the basis where C is diagonal.

(f) Generalise your result in (e) to N levels. That is show that it is possible to diagonalise C in such a way that H becomes block off-diagonal.

Hint: Diagonalise C (you know its eigenvalues!). You can construct H using your result in (e). Think about how to rearrange the rows and columns of your matrices such that the diagonal elements in C are sorted with the positive eigenvalues coming before the negative eigenvalues.