
Quantum Mechanics 2- Problem Set 13

Wintersemester 2016/2017

Abgabe: The problem set will be discussed in the tutorials on **Thursday, 26.01.2017, 09:15**
and **Friday, 27.01.2017, 11:15**

36. Number operator

4 Punkte

Consider an operator \hat{a} which satisfies $\{\hat{a}, \hat{a}^\dagger\} = \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = 1$ and $\{a, a\} = \{a^\dagger, a^\dagger\} = 0$. Show that the operator $\hat{N} = \hat{a}^\dagger\hat{a}$ has eigenvalues 0 and 1. What would you get if the anti-commutator is replaced by a commutator?

37. Tight-binding model

2+2+2+2 Punkte

In this problem we consider a tight-binding model defined on a one-dimensional lattice with N sites. The Hamiltonian can, in second quantised notation, be written as

$$H = -t \sum_i c_{i+1}^\dagger c_i + \text{h.c.},$$

where the sum is over lattice sites $i \in \mathbb{Z}$, c_i^\dagger and c_i are creation and annihilation operators satisfying $[c_i, c_j^\dagger] = \delta_{ij}$, and h.c. stands for hermitian conjugate.

(a) Show that the state

$$|k\rangle = \sum_j e^{ikj} c_j^\dagger |0\rangle,$$

with $|0\rangle$ denoting a state with no particles, is an eigenstate of the Hamiltonian.

(b) Define now

$$c_k = \sum_j e^{-ikj} c_j.$$

Show that $[c_k, c_{k'}^\dagger] = N\delta_{kk'}$. Argue that, for $|k - k'| \ll 1$, the sum can be replaced by an integral, and evaluate the integral for an infinite lattice.

(c) Show that the inverse transformation is

$$c_j = \frac{1}{N} \sum_k e^{ikj} c_k.$$

(d) Show that the Hamiltonian can be written in terms of the new operators as

$$H = \sum_k \epsilon(k) c_k^\dagger c_k,$$

where $\epsilon(k)$ is the spectrum.

38. Berry phase and the Aharonov-Bohm effect *2+2+1+3 Punkte*

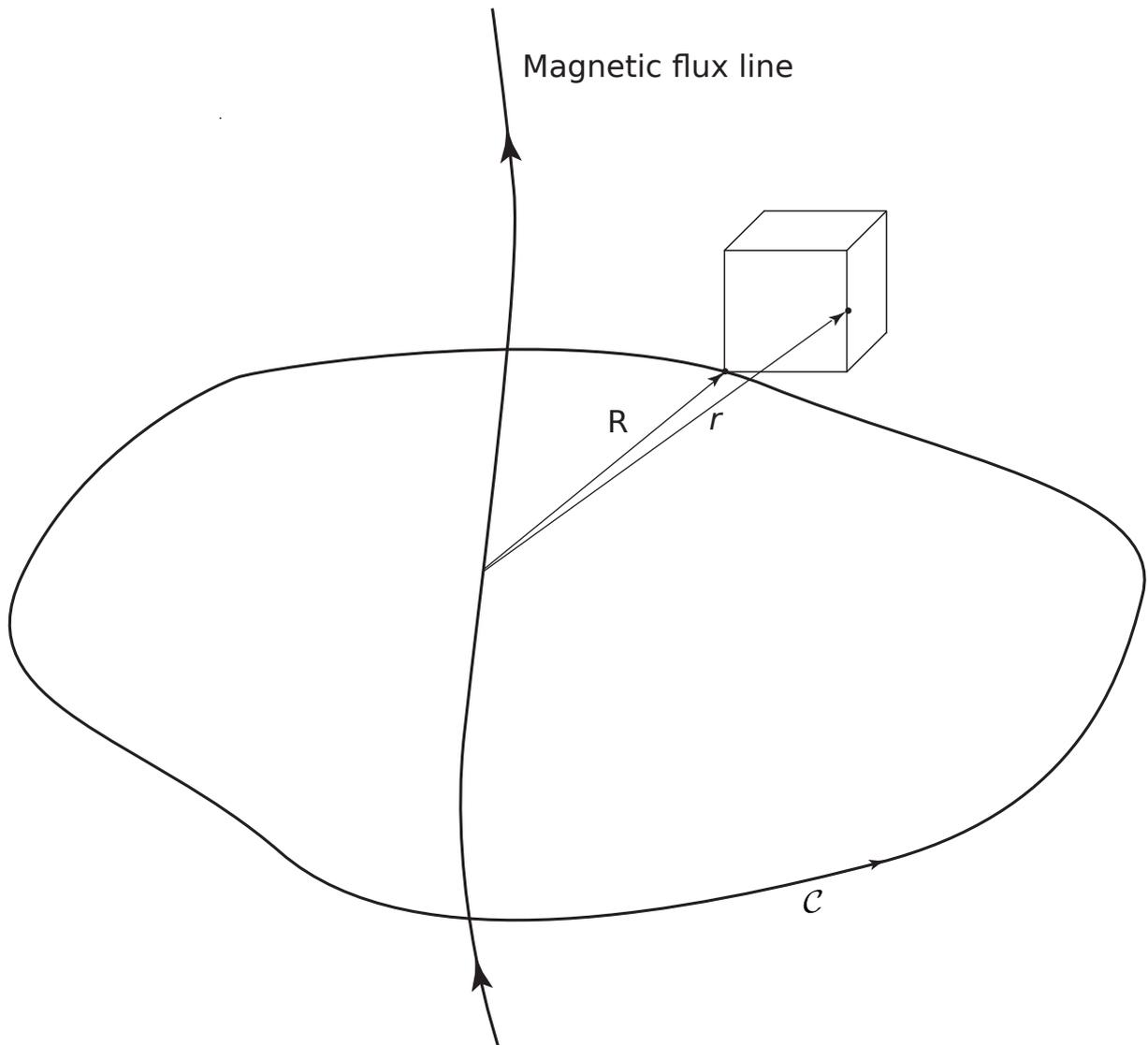


Figure 1: An electron in a box moves around a magnetic flux line. The path of the electron encloses a flux Φ_B .

Consider an electron in a small box moving along a closed loop \mathcal{C} , which encloses a magnetic flux Φ_B as shown in Fig. 1. Let \mathbf{R} denote the position vector of a point on the box and \mathbf{r} the position vector of the electron itself.

- (a) Show that if the wave function of the electron in the absence of a magnetic field is $\psi_n(\mathbf{r})$, then the wave function of the electron in the box at position \mathbf{r} is

$$\langle \mathbf{r} | n(\mathbf{R}) \rangle = \exp \left[\frac{ie}{\hbar} \int_{\mathbf{R}}^{\mathbf{r}} \mathbf{A}(\mathbf{r}') \cdot d\mathbf{r}' \right] \psi_n(\mathbf{r} - \mathbf{R}).$$

Here \mathbf{A} denotes the vector potential.

- (b) Show that

$$\langle n(\mathbf{R}) | \nabla_{\mathbf{R}} | n(\mathbf{R}) \rangle = -\frac{ie}{\hbar} \mathbf{A}(\mathbf{R}).$$

- (c) Calculate the geometric phase

$$\gamma_n(\mathcal{C}) = i \oint_{\mathcal{C}} \langle n(\mathbf{R}) | \nabla_{\mathbf{R}} | n(\mathbf{R}) \rangle \cdot d\mathbf{R},$$

and comment on your result.

- (d) Suppose now an electron moves above or below a very long impenetrable cylinder as shown in the Fig. 2. Inside the cylinder there is a magnetic field parallel to the cylinder axis, taken to be normal to the plane of the figure. Outside the cylinder there is no magnetic field but the particle paths enclose a magnetic flux. Calculate the interference due to the presence of the magnetic flux.

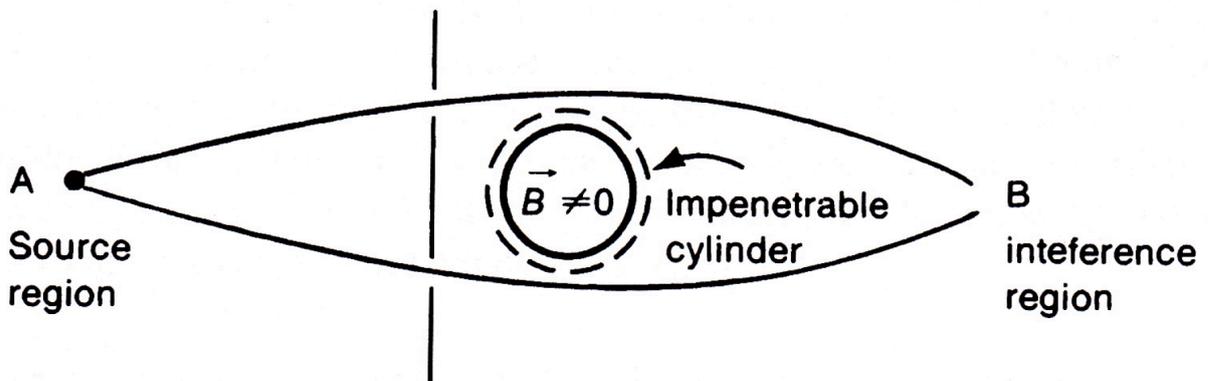


Figure 2: An electron moves either above or below an impenetrable cylinder enclosing a magnetic field parallel to its axis.