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Quantum Mechanics 2- Problem Set 11

Wintersemester 2016/2017

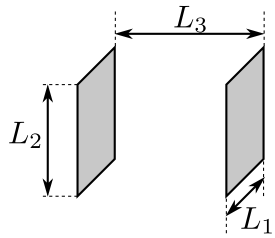
Abgabe: The problem set will be discussed in the tutorials on **Thursday, 12.01.2017, 09:15** and **Friday, 13.01.2017, 11:15**

32. Casimir Effect

4+4+2+1 Punkte

As shown in problem 31, the Hamiltonian of the quantised radiationfield confined to a box with volume $V = L_1L_2L_3$ and with periodic boundary conditions, is given by

$$H = \sum_{\mathbf{k}} \sum_{\lambda=\pm} \hbar\omega_{\mathbf{k}} \big(a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} + \frac{1}{2}\big) , \quad \omega_{\mathbf{k}} = c|\mathbf{k}| , \quad k_i = \frac{2\pi}{L_i} n_i , \quad n_i \in \mathbb{N} .$$



In particular we found that the ground state, in which no modes are excited, has a divergent energy. Whilst this divergent vacuum zero-point energy is not observable, the dependence on the boundaries does lead to observable phenomena.
To investigate this, we consider in the following two conducting plates with surface areas $A = L_1L_2$ seperated by a distance L_3. In the plane of the plates we will still be using periodic boundary conditions and consider the limit $L_1, L_2 \to \infty$. Since the electric field $\mathbf{E}$ between the plates vanishes, only modes with $|\mathbf{E}| \propto \sin(k_3x_3)$ are possible. Here $k_3 = n_3\pi/L_3$ with $n_3 = 1, 2, \ldots$. To get a finite vacuum energy we will moreover introduce an exponential cutoff $e^{-\epsilon\omega_{\mathbf{k}}}$ with $\epsilon > 0$, and take the limit of $\epsilon \to 0$ at the end of the calculation. The energy density per unit plate area between the plates is given by

$$\begin{aligned}\sigma_E(L_3) &= \lim_{L_1,L_2\to\infty} \frac{1}{L_1L_2} \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}}\, e^{-\epsilon\omega_{\mathbf{k}}} \\ &= \hbar c \sum_{n_3=1}^{\infty} \int \frac{d^2k}{(2\pi)^2} \sqrt{k_1^2 + k_2^2 + (\tfrac{\pi n_3}{L_3})^2}\, e^{-\epsilon c\sqrt{k_1^2+k_2^2+(\frac{\pi n_3}{L_3})^2}}\end{aligned}$$

(a) Using polar coordinates and a suitable subsitution show that $\sigma_E(L_3)$ can be written as

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2} \frac{\partial^2}{\partial\epsilon^2} \sum_{n=1}^{\infty} \int_{n\pi c/L_3}^{\infty} d\omega e^{-\epsilon\omega} .$$

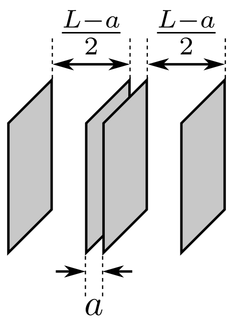
(b) Calculate the integral over ω and perform the sum to show that

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2} \frac{\partial^2}{\partial\epsilon^2} \Big(\frac{1}{\epsilon} \frac{1}{e^{\epsilon\pi c/L_3} - 1}\Big) .$$

Show further that

$$\sigma_E(L_3) = \frac{\hbar}{2\pi c^2}\Big(\frac{6}{\epsilon^4}\frac{L_3}{\pi c} - \frac{1}{\epsilon^3} - \frac{1}{360}\Big(\frac{\pi c}{L_3}\Big)^3 + \mathcal{O}(\epsilon^2)\Big).$$

(c) The energy density calculated in the previous part diverges as the distance between the plates increases ($L_3 \to \infty$). This will be our reference point. We therefore consider two plates separated by a fixed distance a, together with two external plates which are places a further distance $(L-a)/2$ away. The relevant energy density is then given by



$$\sigma_E(a, L) = \sigma_E(a) + 2\sigma_E\left(\frac{L-a}{2}\right).$$

Find an expression for $\sigma_E(a, L)$ using your result in (b).

(d) Since the energy density varies with the distance between plates, the plates experience a pressure which is given by

$$p_{\text{vac}} = -\lim_{L\to\infty}\frac{\partial}{\partial a}\sigma_E(a, L).$$

How large is this pressure for $A = 1\,\text{cm}^2$ and $a = 1\,\mu\text{m}$?

33. Coulomb and Exchange integrals for helium *6+3 Punkte*

The energy of excited states in helium can be shown to be, to leading order in perturbation theory, given by

$$E_{nl,\pm} = -\frac{Z^2}{2}\left(1 + \frac{1}{n}\right) + J_{nl} \pm K_{nl},$$

where the Coulomb- and exchange integrals for helium are given by

$$J_{nl} = \frac{e^2}{4\pi\epsilon_0}\langle u_{100}(\mathbf{r_1})u_{nlm}(\mathbf{r_2})|\frac{1}{|\mathbf{r_1} - \mathbf{r_2}|}|u_{100}(\mathbf{r_1})u_{nlm}(\mathbf{r_2})\rangle,$$

and

$$K_{nl} = \frac{e^2}{4\pi\epsilon_0}\langle u_{100}(\mathbf{r_1})u_{nlm}(\mathbf{r_2})|\frac{1}{|\mathbf{r_1} - \mathbf{r_2}|}|u_{nlm}(\mathbf{r_1})u_{100}(\mathbf{r_2})\rangle,$$

respectively. Here u_{nlm} is the hydrogen wave-function with $Z = 2$.

(a) Calculate the Coulomb- and exchange integrals for $n = 2, l = 0, 1$.

Hint: Express $1/|\mathbf{r_1} - \mathbf{r_2}|$ as a sum over spherical harmonics and use orthogonality of these to perform the angular integrals.

(b) Make a quantitative sketch of the energy levels $E_{nl,\pm}$ of the terms ^{1}S, ^{3}S, ^{1}P, and ^{3}P.

Hint: Recall that the superscript denotes the spin and is given by $2S + 1$ whilst the letter denotes the total orbital angular momentum $L = L_1 + L_2$ ($L = 0$ for S, $L = 1$ for P).