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Quantum Mechanics 2- Problem Set 10

Wintersemester 2016/2017

Abgabe: The problem set will be discussed in the tutorials on **Thursday, 05.01.2017, 09:15** and **Friday, 06.01.2017, 11:15**

29. Addition of angular momenta

3+3+1 Punkte

Consider two angular momenta $\hat{\mathbf{L}}_1$ and $\hat{\mathbf{L}}_2$ with $l_1 = l_2 = 1$. In this problem we will calculate the eigenvalues and eigenfunctions of $\hat{\mathbf{L}}^2$. The eigenfunctions are linear combinations of the 9 functions

$$Y_{1m}(\theta_1, \varphi_1) Y_{1m'}(\theta_2, \varphi_2) = u_m v_{m'}, \qquad \text{with } m, m' = 1, 0, -1.$$

(a) Construct the 9×9 matrix representation of the operator $\hat{\mathbf{L}}^2$ in the $u_m v_{m'}$ basis.

(b) Calculate the eigenvalues of $\hat{\mathbf{L}}^2$ by diagonalising the matrix.

(c) Calculate the corresponding eigenfunctions.

Hint: It is possible to make the matrix block-diagonal, as shown in the figure, by making suitable row- and column-operations.

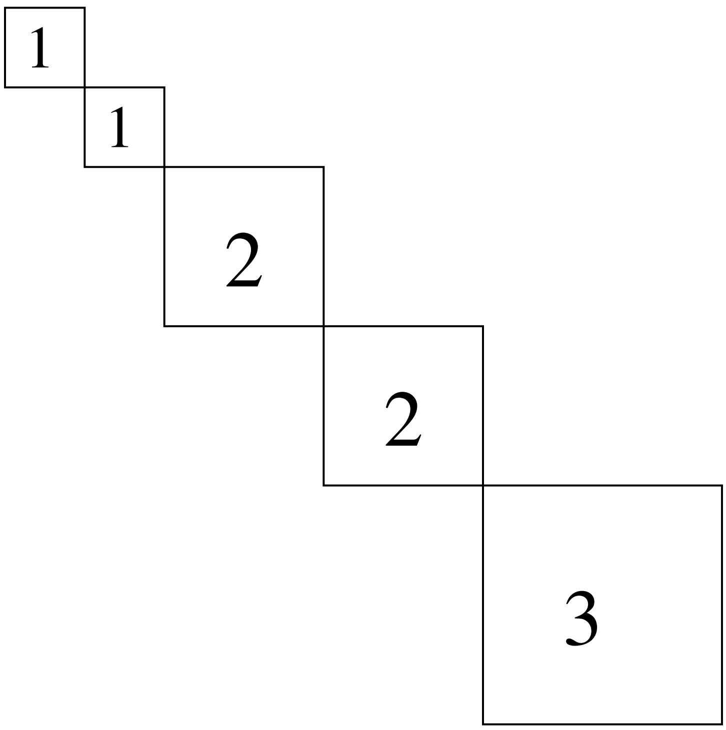

Figure 1: The matrix can be transformed into a block diagonal form.

30. Addition of three angular momenta *3+2 Punkte*

j	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$j_1 + 1$	$\left[\frac{(j_1+m)(j_1+m+1)}{(2j_1+1)(2j_1+2)}\right]^{1/2}$	$\left[\frac{(j_1-m+1)(j_1+m+1)}{(2j_1+1)(j_1+1)}\right]^{1/2}$	$\left[\frac{(j_1-m)(j_1-m+1)}{(2j_1+1)(2j_1+2)}\right]^{1/2}$
j_1	$-\left[\frac{(j_1+m)(j_1-m+1)}{2j_1(j_1+1)}\right]^{1/2}$	$\left[\frac{m^2}{j_1(j_1+1)}\right]^{1/2}$	$\left[\frac{(j_1-m)(j_1+m+1)}{2j_1(j_1+1)}\right]^{1/2}$
$j_1 - 1$	$\left[\frac{(j_1-m)(j_1-m+1)}{2j_1(2j_1+1)}\right]^{1/2}$	$-\left[\frac{(j_1-m)(j_1+m)}{j_1(2j_1+1)}\right]^{1/2}$	$\left[\frac{(j_1+m+1)(j_1+m)}{2j_1(2j_1+1)}\right]^{1/2}$

Figure 2: Table of Clebsch-Gordan coefficients (from B. H. Bransden and C. J. Joachain). The table should be understood as a matrix, as discussed in lectures.

Consider three angular momenta with $l_1 = l_2 = l_3 = 1$.

(a) Add the three angular momenta to get a state with total angular momentum $L = 0$.

Hint: First add L_1 and L_2 and then add the resulting angular momentum with L_3. Use the same basis as in the previous problem, but don't keep all 27 basisfunctions. Instead keep only the ones that have $L = 0$. You can look up the relevant Clebsch-Gordan coefficients in the table.

(b) Show that this state can be written as a 3×3 determinant and that it therefore is anti-symmetric.

31. Quantisation of the Radiation Field *2+3+3 Punkte*

In the absence of charges, and in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the electromagnetic field is described by the Lagrangian

$$L = \frac{1}{2} \int_\Omega d^3x \left[\epsilon_0 \left(\partial_t \mathbf{A}\right)^2 + \frac{1}{\mu_0} \mathbf{A} \nabla^2 \mathbf{A} \right].$$

Here ϵ_0 denotes the vacuum dielectric constant, μ_0 is the vacuum permeability, and Ω is a cuboid with extensions L_x, L_y, and L_z. Note that the speed of light is $c = 1/\sqrt{\epsilon_0 \mu_0}$.

a) Write down the Lagrange equation for $\mathbf{A}$.

b) Find eigenfunctions $\mathbf{A_k}$ and eigenvalues $\omega_{\mathbf{k}}^2$ of the equation

$$-\nabla^2 \mathbf{A} = \frac{\omega_{\mathbf{k}}^2}{c^2}\mathbf{A},$$

by using periodic boundary conditions. It may be useful to introduce, for each $\mathbf{k}$, a set of orthonormal vectors $\{\xi_{\mathbf{k},1}, \xi_{\mathbf{k},2}\}$ which are both perpendicular to $\mathbf{k}$. The time-dependent solution then has a series expansion

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{\Omega}\sum_{\mathbf{k},i} \alpha_{\mathbf{k},i}(t)\mathbf{A}_{\mathbf{k},i}(\mathbf{x}).$$

Insert this series expansion in the Lagrangian, and find the momenta

$$\pi_{\mathbf{k},i} = \frac{\partial L}{\partial \dot{\alpha}_{\mathbf{k},i}},$$

canonically conjugate to the coordinates $\alpha_{\mathbf{k},i}$. Use the Legendre transform $H = \sum_{\mathbf{k},i} \pi_{\mathbf{k},i}\dot{\alpha}_{\mathbf{k},i} - L(\pi_{\mathbf{k},i}, \alpha_{\mathbf{k},i})$ to obtain the Hamiltonian.

c) The classical Hamiltonian $H(\{\pi_{\mathbf{k},i}, \alpha_{\mathbf{k},i}\})$ can be quantised by imposing canonical commutation relations

$$[\alpha_{\mathbf{k},i}, \alpha_{\mathbf{q},j}] = 0, \quad [\pi_{\mathbf{k},i}, \pi_{\mathbf{q},j}] = 0, \quad [\alpha_{\mathbf{k},i}, \pi_{\mathbf{q},j}] = i\hbar\delta_{\mathbf{k},\mathbf{q}}\delta_{i,j},$$

on the coordinates $\alpha_{\mathbf{k},i}$ and their canonically conjugate momenta $\pi_{\mathbf{k},j}$. In analogy to the one-dimensional harmonic oscillator, we define photon creation and annihilation operators

$$a^\dagger_{\mathbf{k},j} = \sqrt{\frac{\epsilon_0\omega_{\mathbf{k}}}{2\hbar}}\left(\alpha_{-\mathbf{k},j} - \frac{i}{\epsilon_0\omega_{\mathbf{k}}}\pi_{\mathbf{k},j}\right), \quad a_{\mathbf{k},j} = \sqrt{\frac{\epsilon_0\omega_{\mathbf{k}}}{2\hbar}}\left(\alpha_{\mathbf{k},j} + \frac{i}{\epsilon_0\omega_{\mathbf{k}}}\pi_{-\mathbf{k},j}\right).$$

Show that $a_{\mathbf{k},j}$ and $a^\dagger_{\mathbf{k},j}$ obey the commutation relations of harmonic oscillator ladder operators, and express the Hamiltonian in terms of $a_{\mathbf{k},j}$ and $a^\dagger_{\mathbf{k},j}$.