Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow
Dr. C. Drukier

Quantum Field Theory of Many-Particle Systems - Problem Set 9

Wintersemester 2015/2016

Abgabe: The problem set will be discussed in the tutorial on **Tuesday, 15.12.2015, 11:00**

Internet: The problem sets can be downloaded from
http://home.uni-leipzig.de/stp/QFT_of_MPS_WS1516.html

16. Lifetime of quasi-particles

2+5 Punkte

The lifetime of quasi-particles can be calculated from the retarded self-energy as

$$\frac{1}{\tau} = -2\mathrm{Im}\Sigma^R(\omega, \mathbf{k_F}).$$

The leading frequency dependence of the self-energy can be computed using the Fock contribution discussed in the last problem sheet. We shall thus take the zero-temperature retarded self-energy to be given by

$$\Sigma^R(\omega, \mathbf{k}) = 2\int_0^\omega \frac{d\epsilon}{2\pi}\int \frac{d^3q}{(2\pi)^3}\frac{V(\epsilon, \mathbf{q})}{\omega - \epsilon - \xi(\mathbf{k}-\mathbf{q}) + i\eta},$$

where $V(\epsilon, \mathbf{q})$ is an interaction between fermions and bosons which in this problem will be taken to be the screened Coulomb interaction. The upper cutoff in the frequency integral is due to the Pauli principle.

a) In lectures it has been shown that the polarization, for small momenta and $|\epsilon| \ll v_F q$, can be written to leading order as $\Pi(\epsilon, q) = \Pi_0 + \Pi_L(\epsilon, q)$, where $\Pi_L(\epsilon, q) = \pi|\epsilon|\rho_F/(2v_F q)$ contains the leading dynamic contribution. Using that $V = V_0/(1-\Pi V_0)$, where $V_0 = 4\pi e^2/q^2$ denotes the bare Coulomb potential, show by expanding to leading order in Π_L that the leading dynamic contribution to the interaction becomes

$$V = V_{\text{scr}}^2\Pi_L(\epsilon, q).$$

In this expression we have defined the static, screened Coulomb potential as $V_{\text{scr}} = V_0/(1 - \Pi_0 V_0)$.

b) Taking the imaginary part before evaluating the integral, and using the identity

$$\lim_{\eta\to 0^+}\mathrm{Im}\frac{1}{x + i\eta} = -\pi\delta(x),$$

calculate the leading frequency dependence of the lifetime of the quasi-particles.
Hint: Start by performing the angular integration. In order to calculate the leading frequency behaviour of the life-time it is sufficient to set all frequencies to zero in the argument of the $\delta-$function (why?). The radial integral can be performed exactly but in order to get an estimate it is also enough to assume that $q \ll \kappa$ where κ is the screening length given by $\kappa^2 = 4\pi e^2\rho_F$.

17. The Cooper Problem *3+3+2 Punkte*

Consider a pair of electrons in a singlet state, described by the symmetric spatial wave function

(1) $$\phi(\mathbf{r}-\mathbf{r}') = \int \frac{d^3k}{(2\pi)^3}\chi(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}.$$

In the momentum representation the Schrödinger equation has the form

(2) $$\left(E - 2\frac{\hbar^2k^2}{2m}\right)\chi(\mathbf{k}) = \int \frac{d^3k}{(2\pi)^3}V(\mathbf{k},\mathbf{k}')\chi(\mathbf{k}').$$

We assume that the two electrons interact in the presence of a degenerate free electron gas, whose existence is felt only via the exclusion principle: electron levels with $k < k_F$ are forbidden to each of the two electrons, which gives the constraint:

(3) $$\chi(\mathbf{k}) = 0, \qquad k < k_F.$$

We take the interaction of the pair to have the simple attractive form

(4) $$V(\mathbf{k_1},\mathbf{k_2}) = \begin{cases} -g, & \epsilon_F \le \frac{\hbar^2k_i^2}{2m} \le \epsilon_F + \hbar\omega_D, \\ 0, & \text{otherwise} \end{cases},$$

with $i = 1, 2$, and look for a bound-state solution to the Schrödinger equation (2) consistent with the constraint (3). Since we are considering only one-electron levels which in the absence of the attraction have energies in the excess of $2\epsilon_F$, a bound state will be one with energy less than $2\epsilon_F$, and the binding energy will be

(5) $$\Delta = 2\epsilon_F - E.$$

a) Show that a bound state of energy E exists provided that

(6) $$1 = g\int_{\epsilon_F}^{\epsilon_F+\hbar\omega_D} d\epsilon \frac{\rho(\epsilon)}{2\epsilon - E},$$

where $\rho(\epsilon)$ is the density of one-electron levels per unit volume for a given spin.

b) Show that Eq. (6) has a solution with $E < 2\epsilon_F$ for arbitrarily weak g, provided that $\rho(\epsilon_F) \neq 0$ and that $\rho(\epsilon)$ is continuous. (Note the crucial role played by the exclusion principle: If the lower cutoff was not ϵ_F, but 0, then since $\rho(0) = 0$, there would *not* be a solution for arbitrarily weak couplings.

c) Assuming that $\rho(\epsilon)$ differs negligibly from $\rho(\epsilon_F) = \rho_F$ in the range $\epsilon_F < \epsilon < \epsilon_F + \hbar\omega_D$, show that the binding energy is given by

(7) $$\Delta = 2\hbar\omega_D \frac{e^{-\frac{2}{g\rho_F}}}{1 - e^{-\frac{2}{g\rho_F}}},$$

or, in the weak coupling limit:

(8) $$\Delta = 2\hbar\omega_D e^{-\frac{2}{g\rho_F}}.$$