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Quantum Field Theory of Many-Particle Systems - Problem Set 4

Wintersemester 2015/2016

Abgabe: The problem set will be discussed in the tutorial on **Tuesday, 10.11.2015, 11:00**

Internet: The problem sets can be downloaded from
http://home.uni-leipzig.de/stp/QFT_of_MPS_WS1516.html

6. Quantum Harmonic Oscillator

3+3+4 Punkte

The harmonic oscillator provides a valuable environment in which the path integral method can be explored. It is one of the few examples, for which the path integral can be evaluated exactly. Its Hamiltonian is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

In this problem we will use the formula

$$\langle x_F|e^{-i\hat{H}t/\hbar}|x_i\rangle = e^{\frac{i}{\hbar}S[x_{cl}]}\int_{r(0)=0,r(t)=0}\mathcal{D}r\, e^{\frac{i}{2\hbar}\int_0^t dt' r(t')\left[-m\partial_{t'}^2-m\omega^2\right]r(t')}, \tag{1}$$

to evaluate the propagator for the quantum harmonic oscillator.

a) Find a solution to the classical equation of motion which satisfies the boundary conditions $x_{cl}(0) = x_i$ and $x_{cl}(t) = x_F$. Use this solution to evaluate the classical action and the exponential prefactor in Eq. (1).

b) Find a Fourier representation for $r(t)$ which satisfies the boundary conditions $r(0) = 0$ and $r(t) = 0$. Use this Fourier representation to determine the eigenvalues λ_n^A of the differential operator $\hat{A}$ in the exponent of the path integral.

c) In order to evaluate the determinant as the products of eigenvalues of $\hat{A}$, use the result for a free particle. Denoting the differential operator for the free particle by $\hat{B}$, the free particle fluctuation factor is

$$F_B = \sqrt{\frac{m}{it2\pi\hbar}}.$$

Use the relation

$$F_A = F_B\prod_{n=1}^{\infty}\left(\frac{\lambda_n^A}{\lambda_n^B}\right)^{-1/2},$$

together with the mathematical identity $z/\sin z = \prod_{n=1}^{\infty}\left(1 - z^2/\pi^2 n^2\right)^{-1}$ to compute the propagator for the quantum harmonic oscillator.

7. Asymptotic series

2+2+1 Punkte

Consider the following integral,

$$I(g) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^2 - gx^4\right], \quad g > 0$$

which can be seen as a caricature of a typical path integral for an interacting many-body system (this is clearly an oversimplified picture because $I(g)$ is only a one-dimensional integral whilst path integrals are infinite-dimensional). If the coupling g is sufficiently small, one can think of expanding $I(g)$ perturbatively in powers of g. The corresponding N-th order approximation reads

$$S_N(g) = \sum_{n=0}^{N} g^n I_n = \sum_{n=0}^{N} \frac{(-g)^n}{n!} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} x^{4n}.$$

a) Calculate the exact expression for the coefficients I_n in the perturbative expansion and, using the Stirling approximation

$$n! \approx n^n e^{-n}, \quad n \gg 1$$

show that for large n

$$g^n I_n \approx (-16gn/e)^n \approx (-16g)^n n!.$$

The above result shows that the perturbative series $\lim_{N\to\infty} S_N(g)$ is divergent for any finite value of g, no matter how small the coupling is. In fact, this behaviour can also be predicted qualitatively noting that $I(g)$ is only defined for $\mathrm{Re}\, g \geq 0$, while it diverges for $\mathrm{Re}\, g < 0$. In other words, the radius of convergence of the series expansion around $g = 0$ is zero.

b) Although diverging in the limit $N \to \infty$, the perturbative expansion $S_N(g)$ can still give an excellent approximation to the exact result $I(g)$ if the series is truncated at a finite N. To verify this statement, use the following inequality,

$$\left| \sum_{n=N+1}^{\infty} \frac{(-a)^n}{n!} \right| \leq \frac{a^{N+1}}{(N+1)!}, \quad a \geq 0$$

in order to find an estimate for the error $\delta_N(g)$ in the N-th order approximation, such that

$$|I(g) - S_N(g)| \leq \delta_N(g).$$

c) Using the expression for I_n obtained in a), calculate for a given value of g the optimal order $N_{\mathrm{opt}}(g)$ in the perturbative expansion, defined as the value of N for which $\delta_N(g)$ reaches its minimum, and find the corresponding error $\delta_{\min}(g) = \delta_{N_{\mathrm{opt}}}(g)$. In particular, show that $\delta_{\min}(g)$ becomes exponentially small in the limit $g \to 0$, thus justifying the applicability of perturbation theory at small coupling. What do you get for $g = 0.001$, 0.01, 0.1, and 1? The perturbative expansion is an example of an *asymptotic series*, i.e. a series expansion of a given function which is formally divergent, but whose finite order truncations become asymptotically close to the exact result in the limit where the expansion parameter vanishes.