
Quantum Field Theory of Many-Particle Systems - Problem Set 3

Wintersemester 2015/2016

Abgabe: The problem set will be discussed in the tutorial on **Tuesday, 03.11.2015, 11:00**

Internet: The problem sets can be downloaded from
http://home.uni-leipzig.de/stp/QFT_of_MPS_WS1516.html

5. Quantisation of the Radiation Field

5+5+5 Punkte

In the absence of charges, and in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the electromagnetic field is described by the Lagrangian

$$L = \frac{1}{2} \int_{\Omega} d^3x \left[\epsilon_0 (\partial_t \mathbf{A})^2 + \frac{1}{\mu_0} \mathbf{A} \nabla^2 \mathbf{A} \right].$$

Here ϵ_0 denotes the vacuum dielectric constant, μ_0 is the vacuum permeability, and Ω is a cuboid with extensions L_x , L_y , and L_z . Note that the speed of light is $c = 1/\sqrt{\epsilon_0 \mu_0}$.

- Derive the equation of motion for \mathbf{A} by extremising the action $S = \int dt L$.
- Find eigenfunctions $\mathbf{A}_{\mathbf{k}}$ and eigenvalues $\omega_{\mathbf{k}}^2$ of the equation

$$-\nabla^2 \mathbf{A} = \frac{\omega_{\mathbf{k}}^2}{c^2} \mathbf{A},$$

by using periodic boundary conditions. It may be useful to introduce, for each \mathbf{k} , a set of orthonormal vectors $\{\xi_{\mathbf{k},1}, \xi_{\mathbf{k},2}\}$ which are both perpendicular to \mathbf{k} . The time-dependent solution then has a series expansion

$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{\Omega} \sum_{\mathbf{k}, i} \alpha_{\mathbf{k}, i}(t) \mathbf{A}_{\mathbf{k}, i}(\mathbf{x}).$$

Insert this series expansion in the Lagrangian, and find the momenta

$$\pi_{\mathbf{k}, i} = \frac{\partial L}{\partial \dot{\alpha}_{\mathbf{k}, i}},$$

canonically conjugate to the coordinates $\alpha_{\mathbf{k}, i}$. Use the Legendre transform $H = \sum_{\mathbf{k}, i} \pi_{\mathbf{k}, i} \dot{\alpha}_{\mathbf{k}, i} - L(\pi_{\mathbf{k}, i}, \alpha_{\mathbf{k}, i})$ to obtain the Hamiltonian.

- c) The classical Hamiltonian $H(\{\pi_{\mathbf{k},i}, \alpha_{\mathbf{k},i}\})$ can be quantised by imposing canonical commutation relations

$$[\alpha_{\mathbf{k},i}, \alpha_{\mathbf{q},j}] = 0, \quad [\pi_{\mathbf{k},i}, \pi_{\mathbf{q},j}] = 0, \quad [\alpha_{\mathbf{k},i}, \pi_{\mathbf{q},j}] = i\hbar\delta_{\mathbf{k},\mathbf{q}}\delta_{i,j},$$

on the coordinates $\alpha_{\mathbf{k},i}$ and their canonically conjugate momenta $\pi_{\mathbf{k},j}$. In analogy to the one-dimensional harmonic oscillator, we define photon creation and annihilation operators

$$a_{\mathbf{k},j}^\dagger = \sqrt{\frac{\epsilon_0\omega_{\mathbf{k}}}{2\hbar}} \left(\alpha_{-\mathbf{k},j} - \frac{i}{\epsilon_0\omega_{\mathbf{k}}} \pi_{\mathbf{k},j} \right), \quad a_{\mathbf{k},j} = \sqrt{\frac{\epsilon_0\omega_{\mathbf{k}}}{2\hbar}} \left(\alpha_{\mathbf{k},j} + \frac{i}{\epsilon_0\omega_{\mathbf{k}}} \pi_{-\mathbf{k},j} \right).$$

Show that $a_{\mathbf{k},j}$ and $a_{\mathbf{k},j}^\dagger$ obey the commutation relations of harmonic oscillator ladder operators, and express the Hamiltonian in terms of $a_{\mathbf{k},j}$ and $a_{\mathbf{k},j}^\dagger$.