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Quantum Field Theory of Many-Particle Systems - Problem Set 11

Wintersemester 2015/2016

Abgabe: The problem set will be discussed in the tutorial on **Tuesday, 12.1.2016, 11:00**

Internet: The problem sets can be downloaded from
http://home.uni-leipzig.de/stp/QFT_of_MPS_WS1516.html

20. Tunneling between metal and superconductor *3+3+3+3 Punkte*

The most important verification of the BCS theory of superconductivity came from electron tunneling experiments, in which the energy gap as a function of temperature was measured and showed excellent agreement with the BCS theory. In this problem electron tunneling between a normal metal and a superconductor will be explored. The notation is the same as in Problem 19. There, we showed that the single particle tunnel current can be expressed as

$$\begin{aligned} I_{\text{single}} &= ie\left[C^{+}_{A,A^\dagger}(-eV) - \left(C^{+}_{A,A^\dagger}(-eV)\right)^*\right] \\ &= -2e\text{Im}\left[C^{+}_{A,A^\dagger}(-eV)\right]. \end{aligned}$$

Here, the Fourier transform

$$C^{+}_{A,A^\dagger}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} C^{+}_{A,A^\dagger}(t),$$

of the retarded correlation function

$$C^{+}_{A,A^\dagger}(t) = -i\Theta(t)\langle[A(t), A(0)]\rangle$$

is used, and the operator A is defined as

$$A = \sum_{\mathbf{k},\mathbf{p}} T_{\mathbf{k},\mathbf{p}} c^\dagger_{\mathbf{k}} c_{\mathbf{p}}.$$

a) We want to obtain the retarded correlation function $C^{+}_{A,A^\dagger}(\omega)$ via analytic continuation from the imaginary time correlation function $C^{+}_{A,A^\dagger}(i\omega_n)$.
Show that

$$C^{\tau}_{A,A^\dagger}(i\omega_n) = \sum_{\mathbf{k},\mathbf{p}} |T_{\mathbf{k},\mathbf{p}}|^2 T \sum_{i\epsilon_l} G_L(i\epsilon_l, \xi_{\mathbf{k}}) G_R(i\epsilon_l - i\omega_n, \xi_{\mathbf{p}}).$$

b) Use the spectral representation

$$G_{L,R}(i\epsilon_l, \xi_{\mathbf{k},\mathbf{p}}) = \int \frac{d\omega}{2\pi} \frac{A(\omega, \xi_{\mathbf{k},\mathbf{p}})}{i\epsilon_l - \omega},$$

for both Green functions to evaluate the Matsubara sum in the expression for $C^{\tau}_{A,A^\dagger}(i\omega_n)$. It will be useful to perform a partial fraction decomposition and to make use of the identity

$$T \sum_{\epsilon_l} \frac{e^{i\eta\epsilon_l}}{i\epsilon_l - \xi_{\mathbf{k}}} = n_F(\xi_{\mathbf{k}}).$$

Analytically continue $i\omega_n \to \omega + i\eta$ to derive the retarded correlation function, and show that the current is given by

$$I = 2e \sum_{\mathbf{k},\mathbf{p}} |T_{\mathbf{k},\mathbf{p}}|^2 \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} A_R(\epsilon, \xi_{\mathbf{k}}) A_L(\epsilon + eV, \xi_{\mathbf{p}})[n_F(\epsilon) - n_F(\epsilon + eV)].$$

c) Now make use of the fact that the tunneling matrix elements can be approximated as being independent of momenta, i.e.

$$T_{\mathbf{k},\mathbf{p}} = T_0,$$

and introduce the tunneling densities of states

$$\rho_L(\epsilon) = \frac{1}{2\pi} \sum_{\mathbf{k}} A_L(\epsilon, \xi_{\mathbf{k}}), \qquad \rho_R(\epsilon) = \frac{1}{2\pi} \sum_{\mathbf{p}} A_R(\epsilon, \xi_{\mathbf{p}}),$$

to show that the tunneling current is given by

$$I_{\text{rmsingle}} = 4\pi e \Omega_L \Omega_R |T_0|^2 \int d\epsilon\, \rho_R(\epsilon) \rho_L(\epsilon + eV)[n_F(\epsilon) - n_F(\epsilon + eV)].$$

Here Ω_L and Ω_R are the volumes of the left and the right system, respectively.

d) Finally, use that for a normal metal $\rho_L(\epsilon) = \rho_F$ and that for the superconductor

$$\rho_R(\epsilon) = \rho_F \frac{|\epsilon|}{\sqrt{\epsilon^2 - |\Delta_0|^2}} \Theta(\epsilon^2 - \Delta^2),$$

to evaluate the current.

21. Kitaev Chain

3+2+2+3 Punkte

In this problem we will discuss a model for a spinless p-wave superconductor. We thus consider a 1D chain of N fermions with periodic boundary conditions. The Hamiltonian is given by

$$H = -\mu \sum_{i=1}^{N} c_i^\dagger c_i - t \sum_{i=1}^{N} \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right) - \Delta \sum_{i=1}^{N} \left(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger \right).$$

a) By Fourier transforming

$$c_m = \frac{1}{\sqrt{N}} \sum_k e^{ikm} \alpha_k,$$

and introducing the Nambu spinors $\Psi^T = \left(\alpha_k, \alpha_{-k}^\dagger \right)$, diagonalise the Hamiltonian and calculate the spectrum.

b) Introduce the two Majorana fermions $\gamma_{A,i}$ and $\gamma_{B,i}$ by writing

$$c_i = \frac{1}{2}(\gamma_{A,i} + i\gamma_{B,i}),$$

with $\gamma_{\beta,i} = \gamma_{\beta,i}^\dagger$ and $\beta = A, B$. Using the fermionic commutation relations

$$\{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0, \qquad \{c_i, c_j^\dagger\} = \delta_{ij},$$

show that the Majorana fermions satisfy

$$\{\gamma_{\beta,i}, \gamma_{\beta',j}\} = 2\delta_{\beta\beta'}\delta_{ij}.$$

c) Show that the Hamiltonian can be written as

$$H = -it \sum_{i=1}^{N-1} \gamma_{A,i}\gamma_{B,i+1},$$

in the case where $\mu = 0$ and $t = \Delta$.

d) Now introduce a new fermionic operator

$$d_i = \frac{1}{2}(\gamma_{A,i} - i\gamma_{B,i+1}), \qquad i = 1, \ldots, N-1.$$

Check that these new operators satisfy the usual fermionic commutation relations and show that the Hamiltonian can be written as

$$H = 2t \sum_{i=1}^{N-1} \left(d_i^\dagger d_i - \frac{1}{2} \right).$$

What does the spectrum of this Hamiltonian look like? What is the degeneracy of the ground state?